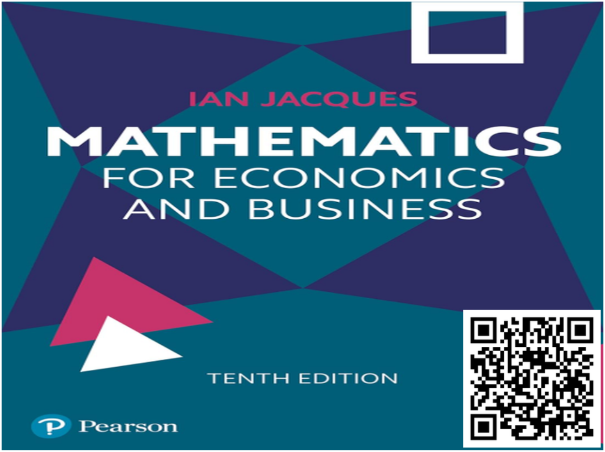
IAN JACQUES
MATHEMATICS
FOR ECONOMICS
AND BUSINESS
TENTH EDITION
Pearson

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To my family: Past, Present and Future

CONTENTS

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PREFACE

This textbook is intended primarily for students on economics, business studies and management courses. It assumes very little prerequisite knowledge, so it can be read by students who have not undertaken a mathematics course for some time. The style is informal, and the text contains a large number of worked examples. Students are encouraged to tackle problems for themselves as they read through each section. Solutions are provided so that all answers can be checked. Consequently, it should be possible to work through this text on a self-study basis. The material is wide ranging and varies from elementary topics such as percentages and linear equations to more sophisticated topics such as constrained optimisation of multivariate functions. The text should therefore be suitable for use on both low- and high-level quantitative methods courses.

This text was first published in 1991. The prime motivation for writing it then was to try to produce a book that students could actually read and understand for themselves. This remains the guiding principle when writing this tenth edition. Extra features and resources requested by students over the years are now incorporated into the printed book. These include end-of-chapter multiple-choice tests and longer examination questions. All of these are designed to help students pass the course.

Online resources include access to the eBook and a bank of new problems in MyLab Math. The latter provides students with an interactive learning environment where they can concentrate on topics that they find difficult and measure their own progress. There is a 'help me solve it' mode which provides step-by-step guidance to solve a problem with the system acting like a personal tutor. Lecturer's resources include fully worked solutions to all of the standard problems in the textbook, PowerPoint slides, and material on more advanced topics.

Many of the examples and questions in this edition have been updated to make them more contemporary. New material on the second-order conditions for Lagrange multipliers and a new section on Lorenz curves and Gini coefficients are provided. Case studies are now included to give an indication how the mathematics covered in each chapter can actually be used by businesses and individuals in practice, which we hope students will find relevant and interesting.

Ian Jacques

INTRODUCTION
Getting Started

NOTES FOR STUDENTS: HOW TO USE THIS TEXT

I am always amazed by the mix of students on first-year economics courses. Some have not acquired any mathematical knowledge beyond elementary algebra (and even that can be of a rather dubious nature), some have never studied economics before in their lives, while others have passed preliminary courses in both. Whatever category you are in, I hope that you will find this text of value. The chapters covering algebraic manipulation, simple calculus, finance, matrices and linear programming should also benefit students on business studies and management courses.

The first few chapters are aimed at complete beginners and students who have not taken mathematics courses for some time. I would like to think that these students once enjoyed mathematics and had every intention of continuing their studies in this area, but somehow never found the time to fit it into an already overcrowded academic timetable. However, I suspect that the reality is rather different. Possibly they hated the subject, could not understand it and dropped it at the earliest opportunity. If you find yourself in this position, you are probably horrified to discover that you must embark on a quantitative methods course with an examination looming on the horizon. However, there is no need to worry. My experience is that every student is capable of passing a mathematics examination. All that is required is a commitment to study and a willingness to suspend any prejudices about the subject gained at school. The fact that you have bothered to buy this text at all suggests that you are prepared to do both.

To help you get the most out of this text, let me compare the working practices of economics and engineering students. The former rarely read individual books in any great depth. They tend to skim through a selection of books in the university library and perform a large number of Internet searches, picking out relevant information. Indeed, the ability to read selectively and to compare various sources of information is an important skill that all arts and social science students must acquire. Engineering students, on the other hand, are more likely to read just a few books in any one year. They read each of these from cover to cover and attempt virtually every problem en route. Even though you are most definitely not an engineer, it is the engineering approach that you need to adopt while studying mathematics. There are several reasons for this. First, a mathematics text can never be described, even by its most ardent admirers, as a good bedtime read. It can take an hour or two of concentrated effort to understand just a few pages of a mathematics text. You are therefore recommended to work through this text systematically in short bursts rather than to attempt to read whole chapters. Each section is designed to take between one and two hours to complete, and this is quite sufficient for a single session. Secondly, mathematics is a hierarchical subject in which one topic follows on from the next. A construction firm building an office block is hardly likely to erect the fiftieth storey without making sure that the intermediate floors and foundations are securely in place. Likewise, you cannot 'dip' into the middle of a mathematics text and expect to follow it unless you have satisfied the prerequisites for that topic. Finally, you actually need to do mathematics yourself before you can understand it. No matter how wonderful your lecturer is, and no matter how many problems are discussed in class, it is only by solving problems yourself that you are ever going to become confident in using and applying mathematical techniques. For this reason, several problems are interspersed within the text, and you are encouraged to tackle these as you go along. You will require writing paper, graph paper, pens and a calculator for this. There is no need to buy an expensive calculator unless you are feeling particularly wealthy at the moment. A bottom-of-the-range **scientific** calculator should be good enough. Answers to every question are printed at the back of this text so that you can check your own answers quickly as you go along. However, please avoid the temptation to look at them until you have made an honest attempt at each one. Remember that in the future you may well have to sit down in an uncomfortable chair, in front of a blank sheet of paper, and be expected to produce solutions to examination questions of a similar type.

At the end of each section there are two parallel exercises. The non-starred exercises are intended for students who are meeting these topics for the first time and the questions are designed to consolidate basic principles. The starred exercises are more challenging but still cover the full range so that students with greater experience will be able to concentrate their efforts on these questions without having to pick-and-mix from both exercises. The chapter dependence is shown in Figure I.1. If you have studied some advanced mathematics before, you will discover that parts of Chapters 1, 2 and 4 are familiar. However, you may find that the sections on economics applications contain new material. You are best advised to test yourself by attempting a selection of problems from the starred exercise in each section to see if you need to read through it as part of a refresher course. Economics students in a desperate hurry to experience the delights of calculus can miss out Chapter 3 without any loss of continuity and move straight on to Chapter 4. The mathematics of finance is probably more relevant to business and accountancy students, although you can always read it later if it is part of your economics syllabus.

At the end of every chapter, you will find a multiple-choice test and some examination questions. These cover the work of the whole chapter. We recommend that you try the multiple-choice questions when you have completed the relevant chapter. As usual, answers

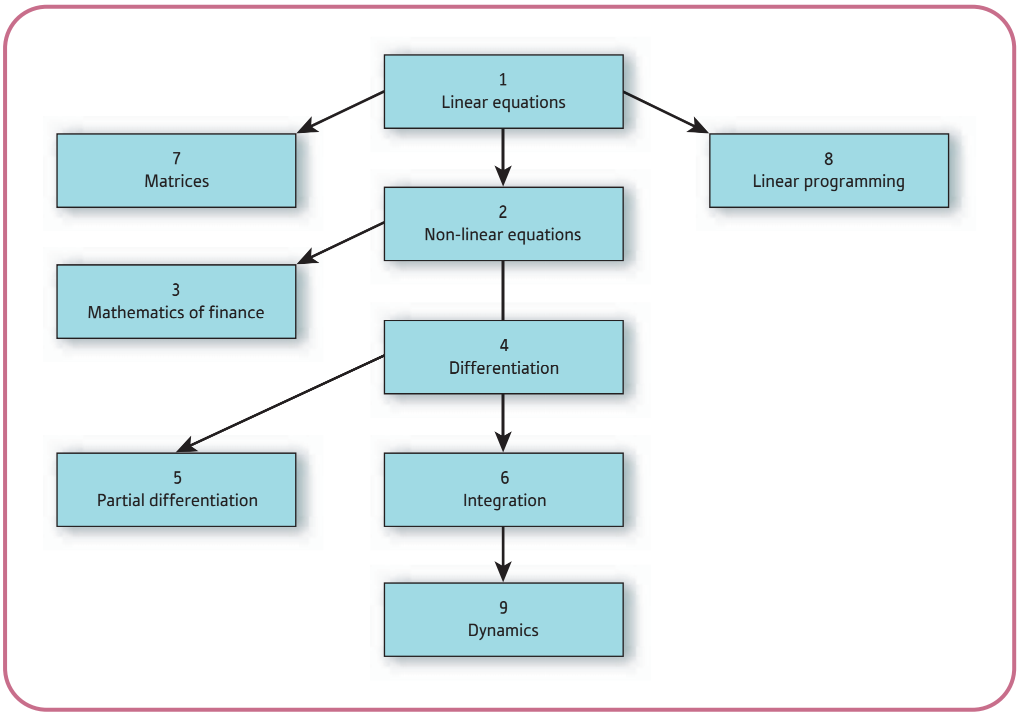


Figure I.1

are provided at the back of the book so that you can check to see how well you have done. If you do get any of the questions wrong, it would be worth redoing them, perhaps writing down full working so that you can spot your mistake more easily. The final section contains several examination-style problems which are more challenging. They tend to be longer than the questions encountered so far in the exercises and require more confidence and experience. You might like to leave these until the end of the course and use them in your build-up to the final exams.

I hope that this text helps you to succeed in your mathematics course. You never know, you might even enjoy it. Remember to wear your engineer's hat while reading the text. I have done my best to make the material as accessible as possible. The rest is up to you!

TOUR OF TEXTBOOK FEATURES

Objectives

Each section begins with a set of bullet points outlining the mathematical skills to be learnt. Use it as a checklist of things that you need to know.

> **Objectives**
>
> At the end of this section you should be able to:
>
> - Add, subtract, multiply and divide negative numbers.
> - Understand what is meant by an algebraic expression.
> - Evaluate algebraic expressions numerically.
> - Simplify algebraic expressions by collecting like terms.
> - Multiply out brackets

Examples

Examples take centre stage in this book and there are lots of them. There is little point in understanding the theory unless you can apply it in practice. All examples are fully worked with every step carefully explained.

> **Example**
>
> (a) Find the value of $2x - 3y$ when $x = 9$ and y
> (b) Find the value of $2Q^2 + 4Q + 150$ when $Q =$
> (c) Find the value of $5a - 2b + c$ when $a = 4$, b
> (d) Find the value of $(12 - t) - (t - 1)$ when $t =$

Practice problems

As their name suggests, these are provided so that you can practise skills that you have just learnt. They are interspersed within the text itself and provide an opportunity to take a break from just passive reading. They help you to consolidate understanding as you go along.

> **Practice Problem**
>
> 1. (1) Without using a calculator, evaluate
>
> (a) $5 \times (-6)$ (b) $(-1) \times (-2)$
>
> (d) $(-5) \div (-1)$ (e) $2 \times (-1) \times (-3)$
>
> (2) Confirm your answer to part (1) using a

Advice

Advice boxes provide some helpful hints and guidance on particular issues and topics.

> **Advice**
>
> In this example the solutions are written out in painstaking de
> how the distributive law is applied. The solutions to all three p
> only one or two steps of working. You are, of course, at liberty
> solutions, but please do not be tempted to overdo this. You m
> later date and may find it difficult if you have tried to be too c

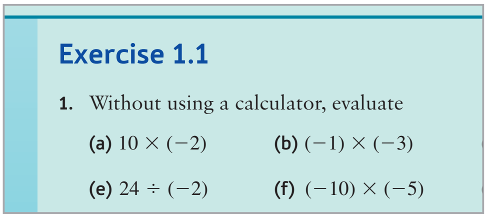
Key terms

Each section concludes with a list of key terms defining all the new mathematical terminology covered in that section. These are collected in a glossary at the end of the book which acts as a kind of dictionary of words and phrases in case you need a quick reminder.

> **Key Terms**
>
> **Autonomous consumption** The level of consumption
> **Autonomous savings** The withdrawals from savings w
> **Consumption function** The relationship between nat
> **Disposable income** Household income after the dedu

Exercises

Every section concludes with a standard exercise. Please remember that you cannot just learn maths by reading books or watching others solve problems. Try as many questions as you can in each exercise, checking your answers with those provided at the back of the book. This is your opportunity to take control of your own learning and boost your confidence.


Exercise 1.1

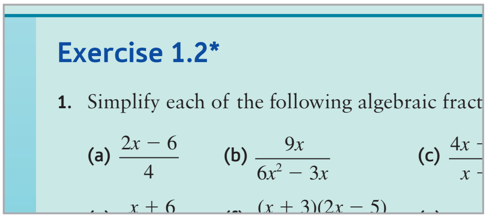
1. Without using a calculator, evaluate

(a) $10 \times (-2)$ (b) $(-1) \times (-3)$

(e) $24 \div (-2)$ (f) $(-10) \times (-5)$

Exercises*

If you find that you are familiar with some of the topics, then you should have a go at the starred exercises instead. They cover the same material as the standard exercise, so you may not need to bother with both. The questions are more challenging, so they should help to maintain your interest if you have covered the topic in a previous life.

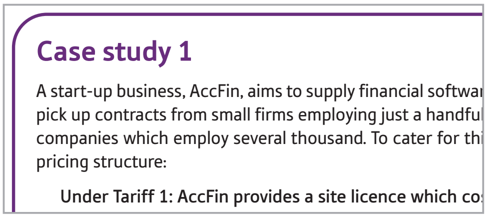

Exercise 1.2*

1. Simplify each of the following algebraic fract

(a) $\frac{2x-6}{4}$ (b) $\frac{9x}{6x^2-3x}$

Case studies

The case studies apply the mathematical techniques to a particular business or personal situation. They show how some of the ideas covered in each chapter can be put into practice.

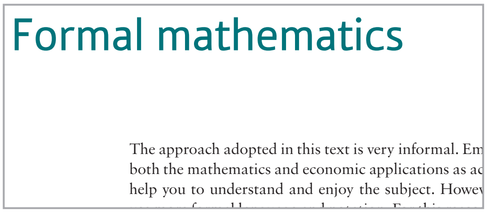

Case study 1

A start-up business, AccFin, aims to supply financial softwa
pick up contracts from small firms employing just a handfu
companies which employ several thousand. To cater for th
pricing structure:

Under Tariff 1: AccFin provides a site licence which co

Formal mathematics

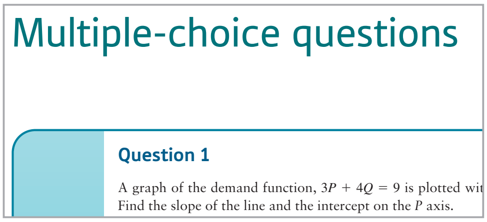
As far as possible this textbook has been written in a relaxed style with the sole purpose of trying to make the mathematics easy to understand. However, much of the mathematics is built upon more theoretical foundations and notation, which are described in these end-of-chapter sections.


Formal mathematics

The approach adopted in this text is very informal. Em
both the mathematics and economic applications as ac
help you to understand and enjoy the subject. Howe

Multiple-choice questions

Once you get to the end of each chapter, you might like to know that you have got the hang of all topics covered. End-of-chapter multiple-choice questions provide a quick way of doing this (without the tedium of having to write out your solutions neatly).

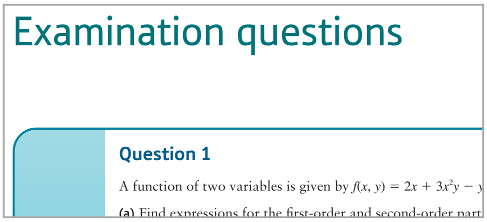

Multiple-choice questions

Question 1

A graph of the demand function, $3P + 4Q = 9$ is plotted wi
Find the slope of the line and the intercept on the P axis.

Examination questions

These are longer questions designed to prepare you for your end-of-year exam. You might like to save them until the end of the course, after you have had a chance to revise properly.


Examination questions

Question 1

A function of two variables is given by $f(x, y) = 2x + 3x^2y -$

CHAPTER 1

Linear Equations

The main aim of this chapter is to introduce the mathematics of linear equations. This is an obvious first choice in an introductory text, since it is an easy topic which has many applications. There are seven sections, which are intended to be read in the order that they appear.

Sections 1.1, 1.2, 1.3, 1.4 and 1.6 are devoted to mathematical methods. They serve to revise the rules of arithmetic and algebra, which you probably met at school but may have forgotten. In particular, the properties of negative numbers and fractions are considered. A reminder is given on how to multiply out brackets and how to manipulate mathematical expressions. You are also shown how to solve simultaneous linear equations. Systems of two equations in two unknowns can be solved using graphs, which are described in Section 1.3. However, the preferred method uses elimination, which is considered in Section 1.4. This algebraic approach has the advantage that it always gives an exact solution and it extends readily to larger systems of equations.

The remaining two sections are reserved for applications in microeconomics and macroeconomics. You may be pleasantly surprised by how much economic theory you can analyse using just the basic mathematical tools considered here. Section 1.5 introduces the fundamental concept of an economic function and describes how to calculate equilibrium prices and quantities in supply and demand theory. Section 1.7 deals with national income determination in simple macroeconomic models.

The first six sections underpin the rest of the text and are essential reading. The final section is not quite as important and can be omitted at this stage.

SECTION 1.1

Introduction to algebra

Objectives

At the end of this section you should be able to:

- Add, subtract, multiply and divide negative numbers.
- Understand what is meant by an algebraic expression.
- Evaluate algebraic expressions numerically.
- Simplify algebraic expressions by collecting like terms.
- Multiply out brackets.
- Factorise algebraic expressions.

ALGEBRA IS BORING

There is no getting away from the fact that algebra *is* boring. Doubtless there are a few enthusiasts who get a kick out of algebraic manipulation, but economics and business students are rarely to be found in this category. Indeed, the mere mention of the word 'algebra' is enough to strike fear into the heart of many a first-year student. Unfortunately, you cannot get very far with mathematics unless you have completely mastered this topic. An apposite analogy is the game of chess. Before you can begin to play a game of chess, it is necessary to go through the tedium of learning the moves of individual pieces. In the same way it is essential that you learn the rules of algebra before you can enjoy the 'game' of mathematics. Of course, just because you know the rules does not mean that you are going to excel at the game, and no one is expecting you to become a grandmaster of mathematics. However, you should at least be able to follow the mathematics presented in economics books and journals as well as to solve simple problems for yourself.

Advice

If you have studied mathematics recently, then you will find the material in the first few sections of the text fairly straightforward. You may prefer just to try the questions in the starred exercise at the end of each section to get yourself back up to speed. However, if it has been some time since you have studied this subject, our advice is very different. Please work through the material thoroughly even if it is vaguely familiar. Make sure that you do the problems as they arise, checking your answers with those provided at the back of this text. The material has been broken down into three subsections:

- negative numbers;
- expressions;
- brackets.

You might like to work through these subsections on separate occasions to enable the ideas to sink in. To rush this topic now is likely to give you only a half-baked understanding, which will result in hours of frustration when you study the later chapters of this text.

1.1.1 Negative numbers

In mathematics, numbers are classified into one of three types: positive, negative or zero. At school you were probably introduced to the idea of a negative number via the temperature on a thermometer scale measured in degrees centigrade. A number such as -5 would then be interpreted as a temperature of 5 degrees below freezing. In personal finance a negative bank balance would indicate that an account is 'in the red' or 'in debit'. Similarly, a firm's profit of $-500\,000$ signifies a loss of half a million.

The rules for the multiplication of negative numbers are

$$\boxed{\text{negative}} \times \boxed{\text{negative}} = \boxed{\text{positive}}$$

$$\boxed{\text{negative}} \times \boxed{\text{positive}} = \boxed{\text{negative}}$$

It does not matter in which order two numbers are multiplied, so

$$\boxed{\text{positive}} \times \boxed{\text{negative}} = \boxed{\text{negative}}$$

These rules produce, respectively,

$$(-2) \times (-3) = 6$$
$$(-4) \times 5 = -20$$
$$7 \times (-5) = -35$$

Also, because division is the same sort of operation as multiplication (it just undoes the result of multiplication and takes you back to where you started), exactly the same rules apply when one number is divided by another. For example,

$$(-15) \div (-3) = 5$$
$$(-16) \div 2 = -8$$
$$2 \div (-4) = -1/2$$

In general, to multiply or divide lots of numbers it is probably simplest to ignore the signs to begin with and just to work the answer out. The final result is negative if the total number of minus signs is odd and positive if the total number is even.

Example

Evaluate

(a) $(-2) \times (-4) \times (-1) \times 2 \times (-1) \times (-3)$ **(b)** $\dfrac{5 \times (-4) \times (-1) \times (-3)}{(-6) \times 2}$

Solution

(a) Ignoring the signs gives

$$2 \times 4 \times 1 \times 2 \times 1 \times 3 = 48$$

There are an odd number of minus signs (in fact, five), so the answer is -48.

(b) Ignoring the signs gives

$$\frac{5 \times 4 \times 1 \times 3}{6 \times 2} = \frac{60}{12} = 5$$

There are an even number of minus signs (in fact, four), so the answer is 5.

Advice

Attempt the following problem yourself both with and without a calculator. On most machines a negative number such as -6 is entered by pressing the button labelled (–) followed by 6.

Practice Problem

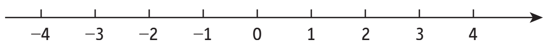
1. **(1)** Without using a calculator, evaluate

(a) $5 \times (-6)$ **(b)** $(-1) \times (-2)$ **(c)** $(-50) \div 10$

(d) $(-5) \div (-1)$ **(e)** $2 \times (-1) \times (-3) \times 6$ **(f)** $\dfrac{2 \times (-1) \times (-3) \times 6}{(-2) \times 3 \times 6}$

(2) Confirm your answer to part (1) using a calculator.

To add or subtract negative numbers it helps to think in terms of a number line:



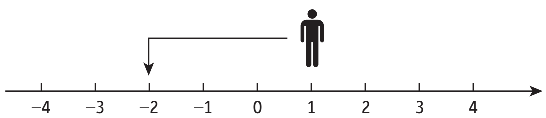
If b is a positive number, then

$$a - b$$

can be thought of as an instruction to start at a and to move b units to the left. For example,

$$1 - 3 = -2$$

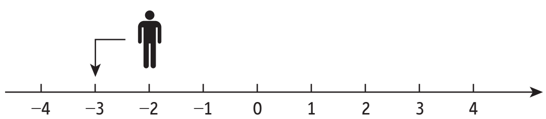
because if you start at 1 and move 3 units to the left, you end up at -2:



Similarly,

$$-2 - 1 = -3$$

because 1 unit to the left of -2 is -3.



On the other hand,

$$a - (-b)$$

is taken to be $a + b$. This follows from the rule for multiplying two negative numbers, since

$$-(-b) = (-1) \times (-b) = b$$

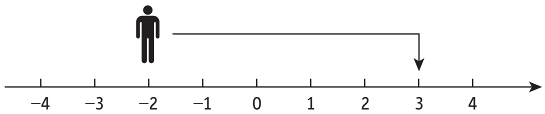
Consequently, to evaluate

$$a - (-b)$$

you start at a and move b units to the right (that is, in the positive direction). For example,

$$-2 - (-5) = -2 + 5 = 3$$

because if you start at −2 and move 5 units to the right, you end up at 3.



Practice Problem

2. **(1)** Without using a calculator, evaluate

(a) $1 - 2$ **(b)** $-3 - 4$ **(c)** $1 - (-4)$

(d) $-1 - (-1)$ **(e)** $-72 - 19$ **(f)** $-53 - (-48)$

(2) Confirm your answer to part (1) using a calculator.

1.1.2 Expressions

In algebra, letters are used to represent numbers. In pure mathematics the most common letters used are x and y. However, in applications it is helpful to choose letters that are more meaningful, so we might use Q for quantity and I for investment. An algebraic expression is then simply a combination of these letters, brackets and other mathematical symbols such as + or −. For example, the expression

$$P\left(1 + \frac{r}{100}\right)^n$$

can be used to work out how money in a savings account grows over a period of time. The letters P, r and n represent the original sum invested (called the principal – hence the use of the letter P), the rate of interest and the number of years, respectively. To work it all out, you not only need to replace these letters by actual numbers, but you also need to understand the various conventions that go with algebraic expressions such as this.

In algebra, when we multiply two numbers represented by letters, we usually suppress the multiplication sign between them. The product of a and b would simply be written as ab without bothering to put the multiplication sign between the symbols. Likewise, when a number represented by the letter Y is doubled, we write $2Y$. In this case we not only suppress the multiplication sign but adopt the convention of writing the number in front of the letter. Here are some further examples:

$P \times Q$ is written as PQ

$d \times 8$ is written as $8d$

$n \times 6 \times t$ is written as $6nt$

$z \times z$ is written as z^2 (using the index 2 to indicate squaring a number)

$1 \times t$ is written as t (since multiplying by 1 does not change a number)

In order to evaluate these expressions it is necessary to be given the numerical value of each letter. Once this has been done, you can work out the final value by performing the operations in the following order:

Brackets first	(B)
Indices second	(I)
Division and Multiplication third	(DM)
Addition and Subtraction fourth	(AS)

This is sometimes remembered using the acronym BIDMAS, and it is essential to use this ordering for working out all mathematical calculations. For example, suppose you wish to evaluate each of the following expressions when $n = 3$:

$2n^2$ and $(2n)^2$

Substituting $n = 3$ into the first expression gives

$2n^2 = 2 \times 3^2$ (the multiplication sign is revealed when we switch from algebra to numbers)

$= 2 \times 9$ (according to BIDMAS, indices are worked out before multiplication)

$= 18$

whereas in the second expression we get

$(2n)^2 = (2 \times 3)^2$ (again, the multiplication sign is revealed)

$= 6^2$ (according to BIDMAS, we evaluate the inside of the brackets first)

$= 36$

The two answers are not the same, so the order indicated by BIDMAS really does matter. Looking at the previous list, notice that there is a tie between multiplication and division for third place, and another tie between addition and subtraction for fourth place. These pairs of operations have equal priority, and under these circumstances you work from left to right when evaluating expressions. For example, substituting $x = 5$ and $y = 4$ in the expression, $x - y + 2$, gives

$x - y + 2 = 5 - 4 + 2$

$= 1 + 2$ (reading from left to right, subtraction comes first)

$= 3$

Example

(a) Find the value of $2x - 3y$ when $x = 9$ and $y = 4$.

(b) Find the value of $2Q^2 + 4Q + 150$ when $Q = 10$.

(c) Find the value of $5a - 2b + c$ when $a = 4$, $b = 6$ and $c = 1$.

(d) Find the value of $(12 - t) - (t - 1)$ when $t = 4$.

Solution

(a) $2x - 3y = 2 \times 9 - 3 \times 4$ (substituting numbers)

$= 18 - 12$ (multiplication has priority over subtraction)

$= 6$

(b) $2Q^2 + 4Q + 150 = 2 \times 10^2 + 4 \times 10 + 150$ (substituting numbers)

$= 2 \times 100 + 4 \times 10 + 150$ (indices have priority over multiplication and addition)

$= 200 + 40 + 150$ (multiplication has priority over addition)

$= 390$

(c) $5a - 2b + c = 5 \times 4 - 2 \times 6 + 1$ (substituting numbers)

$= 20 - 12 + 1$ (multiplication has priority over addition and subtraction)

$= 8 + 1$ (addition and subtraction have equal priority, so work from left to right)

$= 9$

(d) $(12 - t) - (t - 1) = (12 - 4) - (4 - 1)$ (substituting numbers)

$= 8 - 3$ (brackets first)

$= 5$

Practice Problem

3. Evaluate each of the following by replacing the letters by the given numbers:

(a) $2Q + 5$ when $Q = 7$.

(b) $5x^2y$ when $x = 10$ and $y = 3$.

(c) $4d - 3f + 2g$ when $d = 7, f = 2$ and $g = 5$.

(d) $a(b + 2c)$ when $a = 5$, $b = 1$ and $c = 3$.

Like terms are multiples of the same letter (or letters). For example, $2P$, $-34P$ and $0.3P$ are all multiples of P and so are like terms. In the same way, xy, $4xy$ and $69xy$ are all multiples of xy and so are like terms. If an algebraic expression contains like terms which are added or subtracted together, then it can be simplified to produce an equivalent shorter expression.

Example

Simplify each of the following expressions (where possible):

(a) $2a + 5a - 3a$

(b) $4P - 2Q$

(c) $3w + 9w^2 + 2w$

(d) $3xy + 2y^2 + 9x + 4xy - 8x$

Solution

(a) All three are like terms since they are all multiples of a, so the expression can be simplified:

$$2a + 5a - 3a = 4a$$

(b) The terms $4P$ and $2Q$ are unlike because one is a multiple of P and the other is a multiple of Q, so the expression cannot be simplified.

(c) The first and last are like terms since they are both multiples of w, so we can collect these together and write

$$3w + 9w^2 + 2w = 5w + 9w^2$$

This cannot be simplified any further because $5w$ and $9w^2$ are unlike terms.

(d) The terms $3xy$ and $4xy$ are like terms, and $9x$ and $8x$ are also like terms. These pairs can therefore be collected together to give

$$3xy + 2y^2 + 9x + 4xy - 8x = 7xy + 2y^2 + x$$

Notice that we write just x instead of $1x$ and also that no further simplification is possible since the final answer involves three unlike terms.

Practice Problem

4. Simplify each of the following expressions, where possible:

(a) $2x + 6y - x + 3y$ **(b)** $5x + 2y - 5x + 4z$ **(c)** $4Y^2 + 3Y - 43$

(d) $8r^2 + 4s - 6rs - 3s - 3s^2 + 7rs$ **(e)** $2e^2 + 5f - 2e^2 - 9f$ **(f)** $3w + 6W$

(g) $ab - ba$

1.1.3 Brackets

It is useful to be able to take an expression containing brackets and rewrite it as an equivalent expression without brackets, and vice versa. The process of removing brackets is called 'expanding brackets' or 'multiplying out brackets'. This is based on the **distributive law**, which states that for any three numbers a, b and c

$$\boxed{a(b + c) = ab + ac}$$

It is easy to verify this law in simple cases. For example, if $a = 2$, $b = 3$ and $c = 4$, then the left-hand side is

$$2(3 + 4) = 2 \times 7 = 14$$

However,

$$ab = 2 \times 3 = 6 \quad \text{and} \quad ac = 2 \times 4 = 8$$

and so the right-hand side is $6 + 8$, which is also 14.

This law can be used when there are any number of terms inside the brackets. We have

$$a(b + c + d) = ab + ac + ad$$
$$a(b + c + d + e) = ab + ac + ad + ae$$

and so on.

It does not matter in which order two numbers are multiplied, so we also have

$$(b + c)a = ba + ca$$
$$(b + c + d)a = ba + ca + da$$
$$(b + c + d + e)a = ba + ca + da + ea$$

Example

Multiply out the brackets in

(a) $x(x - 2)$

(b) $2(x + y - z) + 3(z + y)$

(c) $x + 3y - (2y + x)$

Solution

(a) The use of the distributive law to multiply out $x(x - 2)$ is straightforward. The x outside the bracket multiplies the x inside to give x^2. The x outside the bracket also multiplies the -2 inside to give $-2x$. Hence

$$x(x - 2) = x^2 - 2x$$

(b) To expand

$$2(x + y - z) + 3(z + y)$$

we need to apply the distributive law twice. We have

$$2(x + y - z) = 2x + 2y - 2z$$
$$3(z + y) = 3z + 3y$$

Adding together gives

$$\begin{aligned} 2(x + y - z) + 3(z + y) &= 2x + 2y - 2z + 3z + 3y \\ &= 2x + 5y + z \qquad \text{(collecting like terms)} \end{aligned}$$

(c) It may not be immediately apparent how to expand

$$x + 3y - (2y + x)$$

However, note that

$$-(2y + x)$$

is the same as

$$(-1)(2y + x)$$

which expands to give

$$(-1)(2y) + (-1)x = -2y - x$$

Hence

$$x + 3y - (2y + x) = x + 3y - 2y - x = y$$

after collecting like terms.

Advice

In this example the solutions are written out in painstaking detail. This is done to show you precisely how the distributive law is applied. The solutions to all three parts could have been written down in only one or two steps of working. You are, of course, at liberty to compress the working in your own solutions, but please do not be tempted to overdo this. You might want to check your answers at a later date and may find it difficult if you have tried to be too clever.

Practice Problem

5. Multiply out the brackets, simplifying your answer as far as possible.

(a) $(5 - 2z)z$ **(b)** $6(x - y) + 3(y - 2x)$ **(c)** $x - y + z - (x^2 + x - y)$

Mathematical formulae provide a precise way of representing calculations that need to be worked out in many business models. However, it is important to realise that these formulae may be valid only for a restricted range of values. Most large companies have a policy to reimburse employees for use of their cars for travel: for the first 50 miles they may be able to claim 90 cents a mile, but this could fall to 60 cents a mile thereafter. If the distance, x miles, is no more than 50 miles, then travel expenses, E (in dollars), could be worked out using the formula $E = 0.9x$. If x exceeds 50 miles, the employee can claim \$0.90 a mile for the first 50 miles but only \$0.60 a mile for the last $(x - 50)$ miles. The total amount is then

$$\begin{aligned} E &= 0.9 \times 50 + 0.6(x - 50) \\ &= 45 + 0.6x - 30 \\ &= 15 + 0.6x \end{aligned}$$

Travel expenses can therefore be worked out using two separate formulae:

- $E = 0.9x$ when x is no more than 50 miles
- $E = 15 + 0.6x$ when x exceeds 50 miles.

Before we leave this topic, a word of warning is in order. Be careful when removing brackets from very simple expressions such as those considered in part (c) in the previous worked example and practice problem. A common mistake is to write

$$(a + b) - (c + d) = a + b - c + d \qquad \text{This is NOT true}$$

The distributive law tells us that the -1 multiplying the second bracket applies to the d as well as the c, so the correct answer has to be

$$(a + b) - (c + d) = a + b - c - d$$

In algebra, it is sometimes useful to reverse the procedure and put the brackets back in. This is called **factorisation**. Consider the expression $12a + 8b$. There are many numbers which divide into both 8 and 12. However, we always choose the biggest number, which is 4 in this case, so we attempt to take the factor of 4 outside the brackets:

$$12a + 8b = 4(? + ?)$$

where each ? indicates a mystery term inside the brackets. We would like 4 multiplied by the first term in the brackets to be $12a$, so we are missing $3a$. Likewise, if we are to generate an $8b$, the second term in the brackets will have to be $2b$.

Hence

$$12a + 8b = 4(3a + 2b)$$

As a check, notice that when you expand the brackets on the right-hand side, you really do get the expression on the left-hand side.

Example

Factorise

(a) $6L - 3L^2$

(b) $5a - 10b + 20c$

Solution

(a) Both terms have a common factor of 3. Also, because $L^2 = L \times L$, both $6L$ and $-3L^2$ have a factor of L. Hence we can take out a common factor of $3L$ altogether.

$$6L - 3L^2 = 3L(2) - 3L(L) = 3L(2 - L)$$

(b) All three terms have a common factor of 5, so we write

$$5a - 10b + 20c = 5(a) - 5(2b) + 5(4c) = 5(a - 2b + 4c)$$

Practice Problem

6. Factorise

(a) $7d + 21$ **(b)** $16w - 20q$ **(c)** $6x - 3y + 9z$ **(d)** $5Q - 10Q^2$

We conclude our discussion of brackets by describing how to multiply two brackets together. In the expression $(a + b)(c + d)$ the two terms a and b must each multiply the single bracket $(c + d)$, so

$$(a + b)(c + d) = a(c + d) + b(c + d)$$

The first term $a(c + d)$ can itself be expanded as $ac + ad$. Likewise, $b(c + d) = bc + bd$. Hence

$$(a + b)(c + d) = ac + ad + bc + bd$$

This procedure then extends to brackets with more than two terms:

$$(a + b)(c + d + e) = a(c + d + e) + b(c + d + e) = ac + ad + ae + bc + bd + be$$

Example

Multiply out the brackets

(a) $(x + 1)(x + 2)$ **(b)** $(x + 5)(x - 5)$ **(c)** $(2x - y)(x + y - 6)$

simplifying your answer as far as possible.

Solution

(a) $(x + 1)(x + 2) = x(x + 2) + (1)(x + 2)$

$= x^2 + 2x + x + 2$

$= x^2 + 3x + 2$

(b) $(x + 5)(x - 5) = x(x - 5) + 5(x - 5)$

$= x^2 - 5x + 5x - 25$

$= x^2 - 25$ the *x*s cancel

(c) $(2x - y)(x + y - 6) = 2x(x + y - 6) - y(x + y - 6)$

$= 2x^2 + 2xy - 12x - yx - y^2 + 6y$

$= 2x^2 + xy - 12x - y^2 + 6y$

Practice Problem

7. Multiply out the brackets.

(a) $(x + 3)(x - 2)$

(b) $(x + y)(x - y)$

(c) $(x + y)(x + y)$

(d) $(5x + 2y)(x - y + 1)$

Looking back at part (b) of the previous worked example, notice that

$$(x + 5)(x - 5) = x^2 - 25 = x^2 - 5^2$$

Quite generally

$$(a + b)(a - b) = a(a - b) + b(a - b)$$

$$= a^2 - ab + ba - b^2$$

$$= a^2 - b^2$$

The result

$$a^2 - b^2 = (a + b)(a - b)$$

is called the **difference of two squares** formula. It provides a quick way of factorising certain expressions.

Example

Factorise the following expressions:

(a) $x^2 - 16$ (b) $9x^2 - 100$

Solution

(a) Noting that

$$x^2 - 16 = x^2 - 4^2$$

we can use the difference of two squares formula to deduce that

$$x^2 - 16 = (x + 4)(x - 4)$$

(b) Noting that

$$9x^2 - 100 = (3x)^2 - (10)^2$$

$(3x)^2 = 3x \times 3x$
$= 9x^2$

we can use the difference of two squares formula to deduce that

$$9x^2 - 100 = (3x + 10)(3x - 10)$$

Practice Problem

8. Factorise the following expressions:

(a) $x^2 - 64$ (b) $4x^2 - 81$

Advice

This completes your first piece of mathematics. We hope that you have not found it quite as bad as you first thought. There now follow a few extra problems to give you more practice. Not only will they help to strengthen your mathematical skills, but also they should improve your overall confidence. Two alternative exercises are available. Exercise 1.1 is suitable for students whose mathematics may be rusty and who need to consolidate their understanding. Exercise 1.1* contains more challenging problems and so is more suitable for those students who have found this section very easy.

If you would like some extra help at any stage of your course, you might like to make use of MyLab Math. This online resource is directly linked to each section of this textbook. It includes a 'Help Me Solve This' option which provides step-by-step guidance on how to tackle each question.

Key Terms

Difference of two squares The algebraic result which states that $a^2 - b^2 = (a + b)(a - b)$.

Distributive law The law of arithmetic which states that $a(b + c) = ab + ac$ for any numbers, a, b, c.

Factorisation The process of writing an expression as a product of simpler expressions using brackets.

Like terms Multiples of the same combination of algebraic symbols.

Exercise 1.1

1. Without using a calculator, evaluate

(a) $10 \times (-2)$ **(b)** $(-1) \times (-3)$ **(c)** $(-8) \div 2$ **(d)** $(-5) \div (-5)$

(e) $24 \div (-2)$ **(f)** $(-10) \times (-5)$ **(g)** $\frac{20}{-4}$ **(h)** $\frac{-27}{-9}$

(i) $(-6) \times 5 \times (-1)$ **(j)** $\frac{2 \times (-6) \times 3}{(-9)}$

2. Without using a calculator, evaluate

(a) $5 - 6$ **(b)** $-1 - 2$ **(c)** $6 - 17$ **(d)** $-7 + 23$

(e) $-7 - (-6)$ **(f)** $-4 - 9$ **(g)** $7 - (-4)$ **(h)** $-9 - (-9)$

(i) $12 - 43$ **(j)** $2 + 6 - 10$

3. Without using a calculator, evaluate

(a) $5 \times 2 - 13$ **(b)** $\frac{-30 - 6}{-18}$ **(c)** $\frac{(-3) \times (-6) \times (-1)}{2 - 3}$ **(d)** $5 \times (1 - 4)$

(e) $1 - 6 \times 7$ **(f)** $-5 + 6 \div 3$ **(g)** $2 \times (-3)^2$ **(h)** $-10 + 2^2$

(i) $(-2)^2 - 5 \times 6 + 1$ **(j)** $\frac{(-4)^2 \times (-3) \times (-1)}{(-2)^3}$

4. The *monthly* running costs of energy for my studio flat vary throughout the year:

Month	Jan–Feb	Mar–Apr	May–Sept	Oct–Nov	Dec
Cost ($)	255	210	90	210	255

I pay by a monthly direct debit of $170 and the statement of my account on 1 January is listed as '−$110', where the '−' sign indicates that I am in debit.

Work out what my statement will show by the end of

(a) September **(b)** December

5. The average monthly temperature in °C for Barrow Point, Alaska, is as follows:

J	F	M	A	M	J	J	A	S	O	N	D
−26	−26	−25	−17	−6	2	5	4	0	−9	−17	−22

Work out the temperature change from

(a) January to June **(b)** April to October **(c)** July to December

Models of climate change predict that these monthly averages will all rise by 3.5°C by the end of this century. What do these models predict the average temperature will be in March 2100?

6. Simplify each of the following algebraic expressions:

(a) $2 \times P \times Q$ **(b)** $I \times 8$ **(c)** $3 \times x \times y$

(d) $4 \times q \times w \times z$ **(e)** $b \times b$ **(f)** $k \times 3 \times k$

7. Simplify the following algebraic expressions by collecting like terms:

(a) $6w - 3w + 12w + 4w$
(b) $6x + 5y - 2x - 12y$
(c) $3a - 2b + 6a - c + 4b - c$
(d) $2x^2 + 4x - x^2 - 2x$
(e) $2cd + 4c - 5dc$
(f) $5st + s^2 - 3ts + t^2 + 9$

8. Without using a calculator, find the value of the following:

(a) $2x - y$ when $x = 7$ and $y = 4$.
(b) $x^2 - 5x + 12$ when $x = 6$.
(c) $2m^3$ when $m = 10$.
(d) $5fg^2 + 2g$ when $f = 2$ and $g = 3$.
(e) $2v + 4w - (4v - 7w)$ when $v = 20$ and $w = 10$.

9. If $x = 2$ and $y = -3$, evaluate

(a) $2x + y$
(b) $x - y$
(c) $3x + 4y$
(d) xy
(e) $5xy$
(f) $4x - 6xy$

10. (a) Without using a calculator, work out the value of $(-4)^2$.

(b) Press the following key sequence on your calculator:

[(−)] [4] [x^2]

Explain carefully why this does not give the same result as part (a) and give an alternative key sequence that *does* give the correct answer.

11. Without using a calculator, work out

(a) $(5 - 2)^2$
(b) $5^2 - 2^2$

Is it true in general that $(a - b)^2 = a^2 - b^2$?

12. Use your calculator to work out the following. Round your answer, if necessary, to two decimal places.

(a) $5.31 \times 8.47 - 1.01^2$
(b) $(8.34 + 2.27)/9.41$
(c) $9.53 - 3.21 + 4.02$
(d) $2.41 \times 0.09 - 1.67 \times 0.03$
(e) $45.76 - (2.55 + 15.83)$
(f) $(3.45 - 5.38)^2$
(g) $4.56(9.02 + 4.73)$
(h) $6.85/(2.59 + 0.28)$

13. Multiply out the brackets:

(a) $7(x - y)$
(b) $3(5x - 2y)$
(c) $4(x + 3)$
(d) $7(3x - 1)$
(e) $3(x + y + z)$
(f) $x(3x - 4)$
(g) $y + 2z - 2(x + 3y - z)$

14. Factorise

(a) $25c + 30$
(b) $9x - 18$
(c) $x^2 + 2x$
(d) $16x - 12y$
(e) $4x^2 - 6xy$
(f) $10d - 15e + 50$

15. Multiply out the brackets:

(a) $(x + 2)(x + 5)$
(b) $(a + 4)(a - 1)$
(c) $(d + 3)(d - 8)$
(d) $(2s + 3)(3s + 7)$
(e) $(2y + 3)(y + 1)$
(f) $(5t + 2)(2t - 7)$
(g) $(3n + 2)(3n - 2)$
(h) $(a - b)(a - b)$

→

16. Simplify the following expressions by collecting together like terms:

(a) $2x + 3y + 4x - y$ (b) $2x^2 - 5x + 9x^2 + 2x - 3$

(c) $5xy + 2x + 9yx$ (d) $7xyz + 3yx - 2zyx + yzx - xy$

(e) $2(5a + b) - 4b$ (f) $5(x - 4y) + 6(2x + 7y)$

(g) $5 - 3(p - 2)$ (h) $x(x - y + 7) + xy + 3x$

17. Use the formula for the difference of two squares to factorise

(a) $x^2 - 4$ (b) $Q^2 - 49$ (c) $x^2 - y^2$ (d) $9x^2 - 100y^2$

18. Simplify the following algebraic expressions:

(a) $3x - 4x^2 - 2 + 5x + 8x^2$ (b) $x(3x + 2) - 3x(x + 5)$

19. A law firm seeks to recruit top-quality experienced lawyers. The total package offered is the sum of three separate components: a basic salary which is 1.2 times the candidate's current salary together with an additional \$3000 for each year worked as a qualified lawyer and an extra \$1000 for every year that they are over the age of 21.

Work out a formula that could be used to calculate the total salary, S, offered to someone who is A years of age, has E years of relevant experience and currently earns \$$N$. Hence work out the salary offered to someone who is 30 years old with five years' experience and who currently earns \$150 000.

20. Write down a formula for each situation:

(a) A plumber has a fixed call-out charge of \$100 and has an hourly rate of \$80. Work out the total charge, C, for a job that takes L hours in which the cost of materials and parts is \$$K$.

(b) An airport currency exchange booth charges a fixed fee of \$10 on all transactions and offers an exchange rate of 1 dollar to 0.8 euros. Work out the total charge, C (in \$) for buying x euros.

(c) A firm provides 5 hours of in-house training for each of its semi-skilled workers and 10 hours of training for each of its skilled workers. Work out the total number of hours, H, if the firm employs a semi-skilled and b skilled workers.

(d) A car hire company charges \$$C$ a day together with an additional \$$c$ per mile. Work out the total charge, \$$X$, for hiring a car for d days and travelling m miles during that time.

Exercise 1.1*

1. Without using a calculator, evaluate

(a) $(12 - 8) - (6 - 5)$ (b) $12 - (8 - 6) - 5$ (c) $12 - 8 - 6 - 5$

2. Put a pair of brackets in the left-hand side of each of the following to give correct statements:

(a) $2 - 7 - 9 + 3 = -17$ (b) $8 - 2 + 3 - 4 = -1$ (c) $7 - 2 - 6 + 10 = 1$

3. Without using a calculator, work out the value of each of the following expressions in the case when $a = 3$, $b = -4$ and $c = -2$:

(a) $a(b - c)$ **(b)** $3c(a + b)$ **(c)** $a^2 + 2b + 3c$ **(d)** $2abc^2$

(e) $\frac{c + b}{2a}$ **(f)** $\sqrt{2(b^2 - c)}$ **(g)** $\frac{b}{2c} - \frac{a}{3b}$ **(h)** $5a - b^3 - 4c^2$

4. Without using a calculator, evaluate each of the following expressions in the case when $x = -1$, $y = -2$ and $z = 3$:

(a) $x^3 + y^2 + z$ **(b)** $\sqrt{\left(\frac{x^2 + y^2 + z}{x^2 + 2xy - z}\right)}$ **(c)** $\frac{xyz(x + z)(z - y)}{(x + y)(x - z)}$

5. Multiply out the brackets and simplify

$$(x - y)(x + y) - (x + 2)(x - y + 3)$$

6. Simplify

(a) $x - y - (y - x)$ **(b)** $(x - ((y - x) - y))$ **(c)** $x + y - (x - y) - (x - (y - x))$

7. Multiply out the brackets:

(a) $(x + 4)(x - 6)$ **(b)** $(2x - 5)(3x - 7)$ **(c)** $2x(3x + y - 2)$

(d) $(3 + g)(4 - 2g + h)$ **(e)** $(2x + y)(1 - x - y)$ **(f)** $(a + b + c)(a - b - c)$

8. Factorise

(a) $9x - 12y$ **(b)** $x^2 - 6x$ **(c)** $10xy + 15x^2$

(d) $3xy^2 - 6x^2y + 12xy$ **(e)** $x^3 - 2x^2$ **(f)** $60x^4y^6 - 15x^2y^4 + 20xy^3$

9. Use the formula for the difference of two squares to factorise

(a) $p^2 - 25$ **(b)** $9c^2 - 64$ **(c)** $32v^2 - 50d^2$ **(d)** $16x^4 - y^4$

10. Evaluate the following without using a calculator:

(a) $50\,563^2 - 49\,437^2$ **(b)** $90^2 - 89.99^2$

(c) $759^2 - 541^2$ **(d)** $123\,456\,789^2 - 123\,456\,788^2$

11. A specialist paint manufacturer receives \$20 for each pot sold. The initial set-up cost for the production run is \$1500 and the cost of making each pot of paint is \$5.

(a) Write down a formula for the total profit, π, if the firm manufactures x pots of paint and sells y pots.

(b) Use your formula to calculate the profit when $x = 1000$ and $y = 800$.

(c) State any restrictions on the variables in the mathematical formula in part (a).

(d) Simplify the formula in the case when the firm sells all that it manufactures.

12. Factorise

(a) $2KL^2 + 4KL$ **(b)** $L^2 - 0.04K^2$ **(c)** $K^2 + 2LK + L^2$

Independent variable A variable whose value determines that of the dependent variable; in $y = f(x)$, the independent variable is x.

Index Alternative word for exponent or power.

Index number The scale factor of a variable measured from the base year multiplied by 100.

Indifference curve A curve indicating all combinations of two goods which give the same level of utility.

Indifference map A diagram showing the graphs of a set of indifference curves. The further the curve is from the origin, the greater the level of utility.

Inelastic demand Where the percentage change in demand is less than the corresponding change in price: $|E| < 1$.

Inferior good A good whose demand decreases as income increases.

Inflation The percentage increase in the level of prices over a 12-month period.

Initial condition The value of Y_0 (or $y(0)$) which needs to be specified to obtain a unique solution of a difference (or differential) equation.

Integer programming A linear programming problem in which the search for solution is restricted to points in the feasible region with whole-number coordinates.

Integral The number $\int_a^b f(x)\mathrm{d}x$ (definite integral) or the function $\int f(x)\mathrm{d}x$ (indefinite integral).

Integration The generic name for the evaluation of definite or indefinite integrals.

Intercept The points where a graph crosses one of the co-ordinate axes.

Internal rate of return (IRR) The interest rate for which the net present value is zero.

Interval The set of all real numbers between (and possibly including) two given numbers.

Inverse (operation) The operation that reverses the effect of a given operation and takes you back to the original. For example, the inverse of halving is doubling.

Inverse function A function, written f^{-1}, which reverses the effect of a given function, f, so that $x = f^{-1}(y)$ when $y = f(x)$.

Inverse matrix A matrix $\mathbf{A}^{-1}$ with the property that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$.

Investment The creation of output not for immediate consumption.

Investment multiplier The number by which you multiply the change in investment to deduce the corresponding change in, say, national income: $\partial Y/\partial I^*$.

IS schedule The equation relating national income and interest rate based on the assumption of equilibrium in the goods market.

Isocost curve A line showing all combinations of two factors which can be bought for a fixed cost.

Isoquant A curve indicating all combinations of two factors which give the same level of output.

L-shaped curve A term used by economists to describe the graph of a function, such as $f(x) = a + \dfrac{b}{x}$, which bends roughly like the letter L.

Labour All forms of human input to the production process.

Labour productivity Average output per worker: Q/L

Lagrange multiplier The number λ which is used in the Lagrangian function. In economics this gives the approximate change in the value of the objective function when the value of the constraint is increased by 1 unit.

Lagrangian function The function $f(x, y) + \lambda[M - \varphi(x, y)]$, where $f(x, y)$ is the objective function and $\varphi(x, y) = M$ is the constraint. The stationary point of this function is the solution of the associated constrained optimisation problem.

Laspeyres index An index number for groups of data which are weighted by the quantities used in the base year.

Law of diminishing marginal productivity (law of diminishing returns) Once the size of the workforce exceeds a particular value, the increase in output due to a 1-unit increase in labour will decline: $\mathrm{d}^2Q/\mathrm{d}L^2 < 0$ for sufficiently large L.

Law of diminishing marginal utility The law which states that the increase in utility due to the consumption of an additional good will eventually decline: $\partial^2U/\partial x_i^2 < 0$ for sufficiently large x_i.

Like terms Multiples of the same combination of algebraic symbols.

Limited growth Used to describe an economic variable which increases over time but which tends to a fixed quantity.

Limits of integration The numbers a and b which appear in the definite integral, $\int_a^b f(x)\mathrm{d}x$.

Linear equation An equation of the form $y = dx + f$.

LM schedule The equation relating national income and interest rate based on the assumption of equilibrium in the money market.

Logarithm The power to which a base must be raised to yield a particular number.

Lorenz curve A graph of $y = L(x)$ where $100y\%$ is the total income that is earned by the bottom $100x\%$ of the population.

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