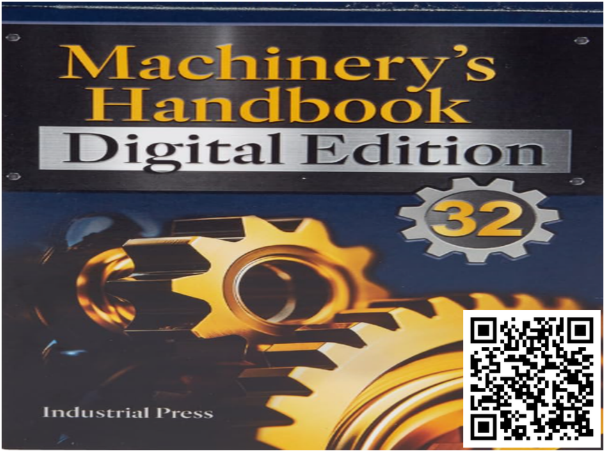
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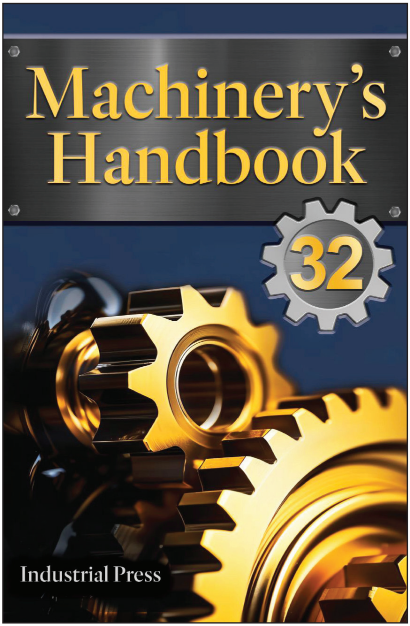

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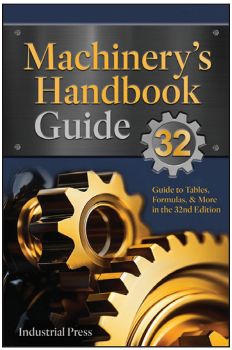

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A REFERENCE BOOK
FOR THE MANUFACTURING AND MECHANICAL ENGINEER,
DESIGNER, DRAFTER, METALWORKER, TOOLMAKER,
MACHINIST, EDUCATOR, STUDENT, AND HOBBYIST

Machinery's Handbook

32nd Edition

BY ERIK OBERG, FRANKLIN D. JONES,
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PREFACE

Industrial Press is celebrating 140 years of excellence and 110 years of continuous publication of the *Machinery's Handbook*. Launched in 1914, this technical masterpiece has since served as the principal reference work in the manufacturing and mechanical industries. Found worldwide in countless engineering and design departments, manufacturing and industrial facilities, machine, metalworking, and other workshops of all sizes, and in general, trade, technical, and engineering schools, colleges, and universities, it is the bestselling and most referenced engineering resource of all time.

Over the *Handbook's* many editions, the editors have strived to carefully craft and perpetually improve a comprehensive, practical resource, covering the most basic yet critical aspects of crucial manufacturing processes. As such, the *Handbook* serves as an invaluable tool, to be ulilized in much the same way as other tools: to design, make, maintain, repair, and otherwise optimize the manufacturing, assembly, and use of parts and machinery of the highest quality, at the lowest cost, as efficiently as possible.

Accordingly, the original edition was designed to fit inside a traditional toolbox. To this day, the compact "toolbox edition" remains the most popular version of the *Handbook*. For those who appreciate the toolbox edition's portability but find the smaller fonts challenging, we now offer it in a bundle with a convenient, flat magnifier.

In 1997, at the request of users, the "desktop" or "large print edition" of the *Handbook* was introduced. The large print version is 40 percent larger than the original toolbox edition, but other than size, the two versions are identical. (The primary typeface in the large print edition is standard reference size, not a large font for visually impaired readers.)

Longtime readers of the *Handbook* will note many changes in recent editions. Yet an enduring goal of the editors is to ensure this encyclopedic reference is as accessible and easy to use as possible. Thus, the *Handbook* continues to incorporate the popular, time-saving thumb tabs. In addition to the table of contents in the front matter, detailed sectional contents begin each major section, and a lengthy index further guides readers.

In 1998, Christopher McCauley launched the *Machinery's Handbook CD-ROM,* containing the contents of the printed book, plus additional indexes and material from earlier editions. Now the *Machinery's Handbook 32 Digital Edition,* this versatile format offers view options, rapid searching, and quick navigation aids in the form of clickable links and cross references to related pages and topics elsewhere in the *Handbook* and in the included *Guide to the Machinery's Handbook*. The growing family of *Machinery's Handbook* products includes printed and digital versions of the *Guide* and the concise *Pocket Companion,* as well as an attractive array of useful combination packages.

As an authoritative reference, the *Machinery's Handbook, 32nd Edition*, may be used to gain vital knowledge of materials, processes, and interrelated considerations; to understand and explore machining and manufacturing options; to reference essential industry data and calculations for part and procedural design and implementation; and to place users on the path to discovering and effectively applying additional data from trusted industry sources. Ultimately, in this age of proliferation of online information, the continued mission of the *Handbook* is to be a carefully curated collection of technical data and information that serves the needs of the manufacturing and mechanical engineering community. To achieve that mission, it must include material of proven and everlasting worth, across the vast spectrum of included topics.

At the same time, the *Machinery's Handbook* (MH) Team editors and world-class experts are faced with the ongoing challenge of selecting suitable material from the enormous body of available data related to traditional yet rapidly evolving fields. The MH Team relies to a great extent on conversations and written communications with users of the *Handbook,* as well as research and comments from subject matter experts. This all-important guidance helps us make difficult decisions about topics and specifics to be introduced, revised, expanded, shortened, or omitted in each new edition.

PREFACE

Thanks to such input and extensive work by the MH Team, this new edition incorporates thousands of individual changes and more than 100 new and revised tables and figures. While we have added a number of pages of expanded and new material to key sections, we also have tightened spacing in many places to prevent the book from growing beyond the physical limits of one volume. In addition, some material excised from the printed books has been moved to the archival collection of legacy material in the *Machinery's Handbook 32 Digital Edition*.

The final result of our concerted efforts, this updated, edited, and reset *Machinery's Handbook, 32nd Edition*, has expanded to 3,008 pages. Among major revisions of existing content and additions of new material are the following:

First and foremost, hundreds of references, explanations, and specifications based on the most current ABMA, ANSI, ASME, ASTM, BS, ISO, SAE, and other industry Standards are updated throughout this edition. Some content changes involved simply verifying and adjusting dates for the latest Standards versions issued by these authoritative institutions. Other changes, such as those derived from the significantly revised ANSI/ASME B18.8.2-2020 Standard, required important data adjustments and procedural updates. These and other vital changes make the *32nd Edition* a must-have resource.

In *PROPERTIES, TREATMENT, AND TESTING OF MATERIALS*, the legacy elements table on page 362 is revised with modern scientific data and is followed by a new table giving an overview of the most commonly used materials. Expanding on the subject of material properties and selection is the timely addition of *Sustainability Considerations for Materials and Processes*, addressing lifecycle, energy, and supply chain; calculating production costs and carbon footprint; and best practices for material recycling and reclamation.

Following industry advances, the *PLASTICS* section, beginning on page 553, also encompasses new and updated information on materials and manufacturing, including the *Rotomolding* production process on page 583.

A new *Polymer Composites* section, on page 585, examines polymer matrices engineered for superior strength and modulus. An associated addition, *CNC Machining of Carbon-Fiber Reinforced Polymers*, on page 1385, covers cutting, drilling, and milling multi-layered matrix materials.

In *DIMENSIONING, GAGING, AND MEASURING*, the instructions and equations for *Tolerance Analysis and Assignment*, beginning on page 685, have been refined. In *MEASURING, INSTRUMENTS, AND INSPECTION METHODS*, the discussion of *Reading Verniers and Micrometers*, beginning on page 691, has been rewritten to facilitate understanding and usage. *MICROMETERS AND CALIPERS*, starting on page 766, also has been reworked with an emphasis on features and use of inch and metric manual and digital devices.

In *MACHINING OPERATIONS*, expert updates can be found throughout the *CNC PROGRAMMING AND CAD/CAM* section, starting on page 1338. Among the expanded and new material is *Shop-Floor Programming*, on page 1341, *Simulation and Verification of CNC Programs*, on page 1384, and *Feature-Based Machining*, on page 1385.

MANUFACTURING PROCESSES also has benefited from additions and refinements. *METALWORKING DIES*, on page 1445, includes new information and figures on classification and types of dies, operations and caveats, and sheet metalworking applications. In *WELDING*, see page 1612 for useful modifications related to positioning of welded joint components.

In *METAL CASTING, MOLDING, AND EXTRUSION*, see *Using Computer Modeling to Optimize Casting Processes*, on page 1518, for more on utilizing this time-saving method for efficient design and production of cast components.

Metal Additive Manufacturing, starting on page 1555, offers an extended list of international *Standards for Additive Manufacturing* that provide guidance for this burgeoning manufacturing segment.

PREFACE

Throughout the *FASTENERS* and *THREADS AND THREADING* sections, specifications have been carefully reviewed and revised, per the current Standards. Among the many updates are specifications for ANSI/ASME, BS, and ISO metric fasteners, starting on page 1713. ISO tables also have been added, including the *Hexalobular Internal Driving Feature* (commonly known as Torx®) table on page 1732.

In *MACHINE ELEMENTS*, coverage of *BALL, ROLLER, AND NEEDLE BEARINGS*, beginning on page 2446, also has been revisited, with new tables, figures, and text on applicable Standards, common configurations, and capabilities of antifriction bearings.

Another section that has received significant revisions is *LUBRICATION*, beginning on page 2502. See page 2526 for *Food-Grade Lubricants* used for machinery and equipment in food and beverage production and processing plants.

In *COUPLINGS, CLUTCHES, BRAKES*, beginning on page 2527, added text and a new table illustrate common types of rigid and flexible couplings.

SHAFT ALIGNMENT, on page 2726, has been revised to include a critical Standard, plus added advice on correcting misalignment. Among other changes, see the *Reverse Dial Alignment Method*, on page 2755; *Laser Alignment* on page 2757; and concluding comments on pros and cons of various alignment methods.

Finally, as mentioned above, the addition of new and revised *Handbook* topics often requires removal of older topics to gain space. Absent in the printed *32nd Edition* are mathematical tables for higher value prime numbers, as well as information on the manufacture of measuring instruments. This extracted material now appears in the *Machinery's Handbook 32 Digital Edition*. Users requiring this information, or wishing to comment on materials removed from this or previous print editions, may reference the *Digital Edition* or contact the editors.

The editors are greatly indebted to all who contact us about possible errors and defects in the current edition of the *Handbook*, offer suggestions for including new or revised material, or pose questions about specific content in the context of modern-day manufacturing problems. Queries involving *Handbook* material usually entail an in-depth review, help us identify subject matter that may require clarification or expansion, and frequently result in improved or new material.

As we constantly seek to improve the *Handbook*, we also welcome new contributors to the team working on each edition, joining the long line of erudite experts who have worked on this master reference. We invite topical experts in industry and academia to reach out to the editors for a discussion of potential future contributions.

Our perpetual goal is to increase the usefulness of the *Handbook* as much as possible. We welcome input and encourage you to send us your thoughts and feedback. The best way to reach us is by emailing MHTeam@industrialpress.com.

ACKNOWLEDGMENTS AND SPECIAL THANKS

The *Machinery's Handbook* is indebted to the whole mechanical field for the data contained in this master reference work. On behalf of the *Handbook* editors past and present, we wish to express our appreciation to all who have assisted us by furnishing data and contributed ideas, corrections, and other commentary.

Most importantly, we thank the thousands of readers who have contacted our team over the years with constructive criticism and suggestions regarding *Handbook* topics and presentation. Your detailed comments, on this edition and about past and future ones, are invaluable.

ACKNOWLEDGMENTS AND SPECIAL THANKS

Many of the American National Standards Institute (ANSI) Standards that deal with mechanical engineering, extracts from which are included in the *Handbook,* are published by the American Society of Mechanical Engineers (ASME). The editors would like to particularly thank those at ASME for their exceptional collaboration in helping us identify and bring essential data up to date, according to the latest, definitive, industry Standards.

While information referencing and discussing various Standards data and nomenclature may be found throughout the *Handbook,* it should be noted that all official Standards and related publications are copyrighted by the issuing industry organizations. For instance, the ASME Standards and all portions and derivative works thereof referenced are the copyrighted property of ASME used under license. Readers should contact ASME directly for further information regarding current editions of the ASME Standards. In fact, the *Handbook* editors always encourage readers to contact the official Standards bodies and to refer to the most up-to-the-minute, complete versions available of all published Standards and related documentation.

On the following pages are brief biographies for the *Machinery's Handbook, 32nd Edition*, Team—an impressive roster of editorial advisory board members and contributors. These esteemed colleagues have played a crucial role in guiding content decisions and advising on specific engineering questions and content challenges. Their lifelong educational and industry experience, impressive technical knowledge and expertise, meticulous research, and exemplary professional collaboration, all have immeasurably enhanced the content of this edition.

We also wish to thank those behind the scenes, our tremendous editorial and production team, without whom this edition would not be possible: the incomparable page layout and production mastermind Jason Hughes; the eagle-eyed trio Stella Bonifazi, Anna Fuller, and Kristen Joseph; and the rest of Scribe's extended editorial and design team; our dedicated team members Susan Gilpin, Gerald Murray, and Rayne Stone; talented cover designer Jeff Weeks; and, as always, our masterful printing and binding partners at Thomson Press.

The *Handbook* is indebted to so many others in industry organizations and associations, educational institutions, commercial enterprises, and private practice. In this context, we cannot thank everyone who has written in, helped resolve a question, or has otherwise spent their valuable time and effort assisting us in improving past and present editions. Therefore, the "Machinery's Handbook Hall of Fame" page—proudly posted in the *Machinery's Handbook Resources* section of our main website, books.industrialpress.com—honors and acclaims these most important participants in the community surrounding this legendary product.

Once again, we look forward to hearing from you and hope that you will share with us how the *Machinery's Handbook* product family informs, supports, and enhances your involvement in this endlessly fascinating field.

Laura Brengelman
Editor

MACHINERY'S HANDBOOK, 32ND EDITION, TEAM

Editorial Advisory Board

Steve Heather, an acclaimed mechanical engineer, worked in the aircraft, automobile, defense, and lighting industries for more than 30 years. More recently, he taught machine shop engineering and computer-aided design (CAD) at the college level to engineering and architectural students. A multi-disciplinary expert, his specialties include AutoCAD®, CAD/CAM, and CNC programming, multistage press and other tool design, and precision machining. He has authored a number of leading technical titles, including *Engineers Precision Data Pocket Reference, AutoCAD® 3D Modeling,* and the *AutoCAD® 2D Resource Kit,* and is coauthor of the multi-book series *Beginning AutoCAD® Exercise Workbook, Advanced AutoCAD® Exercise Workbook,* and *AutoCAD® Pocket Reference,* all published by Industrial Press. An invaluable engineering advisor, exceptional technical consultant, and top contributor of essential material throughout the *Machinery's Handbook,* as well as a highly skilled illustrator, his work enhances numerous pages of the current edition.

David O. Kazmer is Professor and former chair of the Plastics Engineering at the University of Massachusetts Lowell. His teaching and research are focused on product, process, and machine design, as supported by mechanistic and statistical modeling, design for manufacturing, simulation and optimization, and process control techniques. His ongoing work involves developing new manufacturing processes with improved efficiency and performance. A fellow of the American Society of Mechanical Engineers, and a former chair of its Design for Manufacturing and the Life Cycle Technical Committee, he is a fellow of the Society of Plastics Engineers, having served as chair of the Special Interest Group on Process Monitoring and Control. He is the recipient of over twenty recognition awards, an inventor with more than two dozen patents, and author of more than 300 technical publications. His far-reaching knowledge and expert contributions significantly enhance and improve this edition of the *Machinery's Handbook*.

Howard Kuhn most recently served as adjunct professor at the University of Pittsburgh, Swanson School of Engineering, where he taught courses in traditional and additive manufacturing, product realization, and engineering entrepreneurship and performed research on additive manufacturing for tissue engineering. He also has served as a technical advisor to America Makes (National Additive Manufacturing Innovation Institute), where he previously was acting deputy director. Specializing in advanced technology implementation, he has engaged in the design and application of multiple additive manufacturing technologies for government and private industry. He also has developed undergraduate and graduate courses in engineering design, failure analysis, deformation processing, and powder and mechanical metallurgy; written about and conducted tailored training courses on additive manufacturing and more for government agencies, trade organizations, and major companies; and has contributed expert consultation and authoritative revisions to this edition.

Contributors

Vukota Boljanovic received his B.S., M.S., and Ph.D. in mechanical engineering and has nearly 50 years of experience in applied engineering. His impressive career in the aircraft and automotive industries included serving as vice president for research and development with a major aircraft company. He has taught aerospace engineering, among other subjects, and has performed extensive research in development and manufacturing engineering, including aircraft design, assembly, and inspection, the impact of design modification on tooling and production practices, and die design and related material and process selection. He is widely recognized by both academia and industry both for his analytical brilliance and outstanding contributions to manufacturing processes and procedures. In addition to consulting and writing for multiple editions of the *Handbook,* he has authored numerous

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technical papers and titles, including the definitive Industrial Press references *Applied Mathematical and Physical Formulas Pocket Reference; Die Design Fundamentals; Metal Shaping Processes: Casting and Molding, Particulate Processing, Deformation Processes, and Metal Removal; Sheet Metal Forming Processes and Die Design;* and *Sheet Metal Stamping Die Designs: Die Design and Die-Making Practices*.

Ron Brook's career in vibration analysis of rotating machinery started in 1975. His in-depth industry experience includes working with such firms as Nicolet Scientific, Reliance Electric, REM Technologies, Rockwell Automation, and Zonic. He currently serves as Supervisor of Engineering Services, Emeritus, at Integrated Power Services. His years of experience with problem solving and maintenance and reliability challenges have played a part in his acquiring broad technical knowledge and keen subject matter understanding, as well as developing a master set of professional analysis and troubleshooting tools and technical processes for use in the field. This has led to him holding five U.S. patents for homopolar inductive high-speed motors and cracked shaft detection in nuclear reactor coolant pump shafts, as well as his authoring the deep-dive case study, diagnosis, and solutions manual *Rotating Machinery Reliability for Technicians,* published by Industrial Press.

Arief Era received his B.S. and M.S. degrees in mechanical engineering at Columbia University in New York. As a structural analysis engineer, he worked on the designs of commercial aircraft for Boeing. In the field of energy distribution, he has participated in several key projects to improve the aging gas delivery infrastructure serving the greater New York City area. More recently, his efforts in the Energy Efficiency and Demand Management Department of Consolidated Edison of New York have involved implementation of clean energy initiatives, ranging from smart building control systems to heat pump installations. As an exacting, multidisciplinarian, keen-eyed researcher, technical editor, and wordsmith, his countless contributions to this edition—resolving queries from industry experts and hands-on users of the *Machinery's Handbook,* developing, refining, and updating vital information related to industry standards, working on other critical improvements in legacy sections, and more—have been invaluable.

Ken Evans has had an illustrious career working in aerospace and manufacturing as a CNC programmer, innovation center director, prototype lab manager, quality control inspector, technical instructor, sales engineer, machinist, and tool, die, and mold maker. He has more than twenty-five years of experience using and teaching about CAD/CAM programing and processes. In addition to widespread, expert industry consulting, he has served as advisor of multiple medalists in the Precision Machining Technology area at SKILLS USA. He has published several top-selling titles with Industrial Press, including the *CNC Machining Certification Exam Guide: Setup, Operation, Programming; Programming of CNC Machines* and its accompanying *Student Workbook;* and *Interpretation of Geometric Dimensioning and Tolerancing*.

Charles Gillis received his B.S. in mechanical engineering from Worcester Polytechnic Institute and his M.S. in mechanical engineering from Northeastern University, and has nearly 30 years of machine design experience. He previously served as a mechanical engineer for the Gillette Company, designing automated machinery for manufacturing blade and razor products. Through his ongoing work with Dynamic Design Consulting, he helps clients with product development and documentation, design for assemblability and manufacturability, and tolerance stack-up analysis. In addition to being a licensed P.E., he holds a Geometric Dimensioning and Tolerancing Professional Certificate–Senior Level (GDTP-S) from ASME. He has been training engineers in geometric dimensioning and tolerancing, print reading, and related mechanical design and documentation topics for the last decade and a half. Author of the bestselling *Hammer's Blueprint Reading Basics* and *Blueprint Reading Basics Instructor's Resource Kit,* he also has contributed to other Industrial Press titles, including *Cam Design and Manufacturing Handbook* and *Machine Designers Reference*.

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Michael D. Holloway earned a B.S. in chemistry and a B.A. in philosophy from Salve Regina University, and an M.S. in polymer engineering from the University of Massachusetts. He has held positions as a research engineering and product development chemist, applications engineering specialist, aerospace materials and polymer development engineer, and director of technical development and reliability with such firms as ALS Tribology, General Electric Plastics, NCH Corporation, Olin Chemical, Rohm & Haas Electronic Chemicals, and W.R. Grace. Recently, he joined SGS (Société Générale de Surveillance) of Geneva, Switzerland, as global technical manager. Among his specialties, he has focused on lubricants used in heavy industrial equipment, mining, and construction. Known for conducting on-site seminars, professional training, and speaking at industry events, he has served as North Texas section chair for the Society of Tribologists and Lubrication Engineers, national section chair for the International Council for Machinery Lubrication, and adjunct professor at the University of North Texas, and holds multiple professional certifications. He has authored more than a dozen books, including the *Machinery Lubrication Technician (MLT) I and II Certification Exam Guide,* published by Industrial Press.

Krishna Murty Kommajosyula draws on more than 50 years of successful industry experience. Upon retiring from Coromandel International, after 38 years in maintenance management, he was asked to return and served as head of technical training for another 9 years. As a leading maintenance and reliability management expert and root cause failure analyst, he has provided detailed assessments and process improvement recommendations for power, metal, cement, sugar, and petrochemical industries throughout India and has worked with Dr. Reddy's Laboratories as an engineering consultant. With his extensive experience and broad expertise, he is a highly sought-after technical trainer, renowned for his popular science books, electronics and science magazine articles, and radio broadcasts. He also has authored a number of bestselling titles, including the Industrial Press classic *All-in-One Manual of Piping Practice: On-The-Job Solutions, Tips and Insights* and his more recent practical guide *Why Industrial Bearings Fail: Analysis, Maintenance, and Prevention*.

David R. Quinonez has over 25 years of experience in welding, welding inspection, and nondestructive testing (NDT). His impressive experience in NDT began with nuclear submarines and aircraft carriers. Subsequent positions included NDT, welding inspection, and other work with Rolls-Royce gas turbine engines, Lockheed Martin F-22 stealth fighter airframes, and a wide range of missile defense, defense/commercial rocket, pipeline, and structural steel projects for the private sector and public works. A Certified Welding Inspector, he regularly performs visual welding inspection, detailed dimensional verification, and NDT procedures. His Level II certifications include MT, PT, and UT; past certifications have included Level II RT, ET, and ASNT Level III MT and PT. As an expert in welding processes, he has authored the highly effective teaching and learning text *1,001 Questions & Answers for the CWI Exam: Welding Metallurgy and Visual Inspection Study Guide,* published by Industrial Press.

Richard Rex has worked on lathes and milling machines for decades, starting in his teen years in a home workshop and later working with various production machines. He still regularly uses his home machine shop. For 10 years, he worked on product development with Hewlett-Packard and Brown Boveri in the United Kingdom. As chief executive officer of several manufacturing companies in the United States, he established a number of engineering model labs with the usual complement of lathes, milling machines, and metrology equipment. In recent years, he has provided expert consultation and written numerous manuals and technical bulletins for machine tool developers and suppliers. Drawing on his wealth of experience in the metalworking industry, he recently authored two excellent instructional guides, including useful information for metal shops and operators at every level, *Choosing & Using the Right Metal Shop Lathe* and *Choosing & Using the Right Milling Machine,* both published by Industrial Press.

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REAL NUMBERS AND THEIR OPERATIONS

Real Numbers

Most mathematical computation is performed in the *real number system*. The universal set of the "reals" includes the subsets: *naturals, whole numbers, integers, rationals,* and *irrationals*. The naturals (also called *counting numbers*): {1, 2, 3, . . .} are included in the whole numbers: {0, 1, 2, 3, . . .}, which are included in the integers (or signed whole numbers): {. . . ,–2,–1,0,1,2,. . .}. And all of these subsets are included in the rationals.

Rational numbers, including integers, can be written in fraction form. Since all fractions can be divided numerator by denominator, their decimal form either terminates or repeats. Examples of rational numbers: –4/1, 3/5 = 0.6, 1/3 = 0.333. . . .

The only set in the real numbers larger than the naturals that does not contain any of the other sets is the irrationals. These are *not* expressible as ratios. An irrational number's decimal representation does not terminate and it has no pattern of repetition. Examples of irrational numbers are roots that cannot be simplified, such as $\sqrt{6}$ and $\sqrt[3]{70}$, as well as quantities like π and the natural log base *e*. The entire real number set is the union of the rationals and the irrationals.

Properties of Real Numbers.—Though often obvious and followed almost automatically, the properties of real numbers are critical to mathematical reasoning. These properties justify various steps in solving algebraic problems, such as those in this *Handbook*. Equivalence properties (symmetry, reflexivity, transitivity) and operational properties of numbers are summarized here.

Equivalence Properties: The properties of equivalence relations are the basis of equation solving.

Reflexive: $a = a$.
Symmetric: If $a = b$, then $b = a$.
Transitive: If $a = b$ and $b = c$, then $a = c$.
Substitution: If $a = b$, then a may be replaced by b in any equation or expression.

Operational Properties: These concern addition, subtraction, multiplication, and division, as summarized in the table below.

Property	*Addition*	*Multiplication*
Commutative:	$a + b = b + a$	$a \times b = b \times a$
Associative:	$(a + b) + c = a + (b + c)$	$(a \times b) \times c = a \times (b \times c)$
Identity:	$a + 0 = 0 + a = a$	$1 \times a = a \times 1 = a$
Inverse:	$a + (-a) = 0$	$a \times 1/a = 1$

Other Properties:

Distributive of multiplication over addition:	$a \times (b + c) = (a \times b) + (a \times c)$	$(a + b) \times c = (a \times c) + (b \times c)$
Zero property of multiplication:	If $a \times b = 0$, then either $a = 0$ or $b = 0$	
Zero property of division:	If $a/b = 0$, then $a = 0$ $(b \neq 0)$	

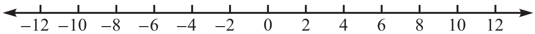
Integers (Signed Numbers).—Positive whole numbers extend to the right of zero on the number line. Negative whole numbers extend to the left of zero. Together with zero, these make up the *integers* (sometimes called *signed numbers*): {. . . ,–2,–1,0,1,2,. . .}.

The sciences (as well as economics and other fields) deal with negative as well as non-negative quantities. Temperature is an obvious example; so is land altitude, which can be above, at, or below sea level. Angles can be negative, too, as explained in

TRIGONOMETRY: SOLUTION OF TRIANGLES. Calculators facilitate computation that involves integers (signed numbers). Knowing the rules of integer operations prevents errors that might occur when a calculator is used.

Absolute Value: A number's absolute value, sometimes called its *magnitude*, is the number's distance from zero on the number line. Whether a number is positive or negative, its absolute value is positive. For example, the absolute value of both 5 and –5 is 5. The absolute value of n is notated $|n|$; thus, $|5| = 5$ and $|-5| = 5$. Absolute value helps explain the rules of signed number addition and subtraction.

Real Number Line: The real number line is generally shown with only the integers marked off (though all numbers are included). A number line is useful for conveying how signed numbers are added or subtracted.



Operations on Signed Numbers: The following rules of operations apply to rational and irrational numbers as well. For simplicity, only integers are given as examples.

Addition and Subtraction: Adding a negative number is equivalent to subtracting its absolute value. When a larger number is subtracted from a smaller number, the result is negative. The rules for adding and subtracting integers are illustrated with an example using four values: 7, 11, –7, and –11. The following examples illustrate the rules:

Examples, Addition

$$7 + 11 = 18$$
$$7 + (-11) = 7 - 11 = -4$$
$$(-7) + 11 = 11 + (-7) = 11 - 7 = 4$$
$$(-7) + (-11) = -18$$

Examples, Subtraction

$$7 - 11 = -4$$
$$7 - (-11) = 7 + 11 = 18$$
$$(-7) - (-11) = (-7) + 11 = 11 + (-7) = 11 - 7 = 4$$
$$-7 - 11 = -18$$

Multiplication and Division: Multiplication or division of numbers with the same sign results in a positive answer. Opposite signed numbers result in negative answers when multiplied or divided. The following examples illustrate the rules:

Examples, Multiplication

$$5 \times 2 = 10$$
$$5 \times (-2) = -10$$
$$(-5) \times 2 = -10$$
$$(-5) \times (-2) = 10$$

Examples, Division

$$12 \div 3 = 4$$
$$(-12) \div 3 = -4$$
$$(12) \div (-3) = -4$$
$$(-12) \div (-3) = 4$$

Order of Operations.—Mathematical operations are performed on numbers in a particular order, commonly referred to as PEMDAS, which stands for "**P**arentheses, **E**xponents, **M**ultiplication, **D**ivision, **A**ddition, **S**ubtraction." First, when there are no parentheses or other grouping symbols, multiplication and division are done before addition and subtraction. Then, proceeding from left to right, the addition and subtraction are done in the order they appear. For example:

$$100 - 26 + 7 \times 2 - 100 \div 4 = 100 - 26 + 14 - 25 = 74 + 14 - 25 = 88 - 25 = 63$$

Parentheses ( ) and brackets [ ]—called *grouping symbols*—indicate if addition and subtraction are to occur before multiplication and division. The operations are performed from the innermost to the outermost grouping symbols. For example:

$$[6 \times (15 - 7)] \div 2 = [6 \times 8] \div 2 = 48 \div 2 = 24$$

Exponents are a multiplication operation, but unless parentheses or brackets are present, exponents are applied before multiplication. For example:

$$4 \times 9^2 = 4 \times 81 = 324$$

Also, when parentheses are present next to a multiplication, the × can be omitted:

$$5(8 - 3) = 5(5) = 25$$

As explained in *Fractions*, the horizontal line in a fraction implies division. The top number (called the *numerator*) is divided by the bottom number (called the *denominator*). For example,

$$\frac{50}{10} = 50 \div 10 = 5$$

In formulas, the multiplication sign (×) may be omitted (when letters—called variables—are multiplied) or replaced by parentheses, which serve the same purpose.

$$A \times B = AB, \qquad 6 \times 4 = (6)(4), \qquad 8 \times a = 8a$$

A multiplication dot (·) is also sometimes used.

Fractions

Rational numbers can be written as *common fractions* or as *decimal fractions*. Common fractions are written as $\frac{a}{b}$ or a/b, where a (the numerator) and b (the denominator) are integers (but b cannot be 0, since division by zero is not defined). The denominator represents the number of equal parts that a whole quantity is broken into. The numerator is the number of these parts under consideration. For example, $\frac{2}{5}$ indicates the whole of something is broken into 5 equal parts, and 2 of these parts are being considered. Any integer is a fraction with a denominator of 1. For example, $\frac{6}{1} = 6$.

The implied operation in a fraction is division. Thus, $\frac{a}{b}$ means $a \div b$.

Multiple: A multiple of a number n is the result of multiplying n by positive integer 1, 2, 3, . . . Thus, the multiples of 3 are 3, 6, 9, 12, . . . The *least common multiple* (LCM) of two or more numbers is the smallest multiple the numbers have in common. In the example below, the first few multiples of 6 and 20 are shown, with the LCM indicated in bold:

6: 6, 12, 18, 24, 30, 36, 42, 48, 54, **60**, 66, . . .
20: 20, 40, **60**, 80, . . .

Thus, 60 is the LCM of 6 and 20.

Factor: An integer a is a factor of n if there is no remainder when n is divided by a. That is, if the result of $n \div a$ is an integer. For example, 3 is a factor of 12 because $^{12}/_{3} = 4$. The greatest common factor (GCF) of two or more numbers is the largest of their common factors. Thus, the common factors of 12 and 18 are 2, 3, and 6; 6 is the GCF.

Unit Fraction: A fraction having the same numerator and denominator is the unit fraction, 1 (or "one whole"). For example, $^{2}/_{2}$, $^{4}/_{4}$, $^{8}/_{8}$, $^{16}/_{16}$, $^{32}/_{32}$, and $^{64}/_{64}$ all equal 1.

Proper Fraction: A fraction whose numerator is less than its denominator. $^{1}/_{4}$, $^{1}/_{2}$, and $^{47}/_{64}$ are examples of proper fractions. The value of any proper fraction is less than 1.

Improper Fraction: A fraction whose numerator is greater than its denominator. $^{3}/_{2}$, $^{5}/_{4}$, and $^{-17}/_{8}$ are examples of improper fractions. The absolute value of any improper fraction is greater than 1.

Reducible Fraction: A reducible fraction is a common fraction in which numerator and denominator have a common factor and so can be reduced to lowest terms by dividing both numerator and denominator by this common factor. For example, in the fraction $^{12}/_{18}$, the numerator and denominator have a GCF of 6. Thus, $^{12}/_{18}$ reduces to $^{2}/_{3}$ by dividing each part of the fraction by 6. A fraction such as $^{16}/_{21}$ cannot be reduced, since 16 and 21 do not have a common factor.

Mixed Number: A mixed number is a combination of a whole number and a proper fraction. The implied operation between them is addition. For example, $4\frac{2}{9}$ means $4 + \frac{2}{9}$. A mixed number is converted to an improper fraction by multiplying the whole number part with the denominator and adding the numerator to obtain the numerator of the final fraction; the denominator remains the same.

Examples:

$$5\frac{2}{3} = 5 + \frac{2}{3} = \frac{15}{3} + \frac{2}{3} = \frac{17}{3} \qquad 9\frac{1}{2} = 9 + \frac{1}{2} = \frac{18}{2} + \frac{1}{2} = \frac{19}{2}$$

To convert mixed numbers to improper fractions, multiply the whole number by the denominator and add the numerator to obtain the new numerator. The denominator remains the same. For example,

$$2\frac{1}{2} = \frac{2 \times 2 + 1}{2} = \frac{5}{2}$$

$$3\frac{7}{16} = \frac{3 \times 16 + 7}{16} = \frac{55}{16}$$

An improper fraction is converted to its mixed number form by dividing the numerator by denominator and placing the remainder over the denominator. Sometimes the fraction part can be reduced, as the second example shows.

$$\frac{17}{8} = 17 \div 8 = 2\frac{1}{8}$$

$$\frac{26}{16} = 26 \div 16 = 1\frac{10}{16} = 1\frac{5}{8}$$

Equivalent Fractions: A fraction raised to its equivalent form ("higher terms") by multiplying numerator and denominator by the same number (that is, by multiplying by a form of 1). For example, $\frac{1}{4} \times \frac{4}{4} = \frac{4}{16}$ and $\frac{3}{8} \times \frac{4}{4} = \frac{12}{32}$.

Any integer *n* can be expressed as a fraction with a chosen denominator value of *m* by simply writing *n* as $n/1$ and multiplying by m/m.

Example: To express 4 as an equivalent fraction with a denominator of 16, write $\frac{4}{1} \times \frac{16}{16} = \frac{64}{16}$

Reciprocal: The *reciprocal* of any number *a* other than 0 is $1/a$. (0 has no reciprocal, since 1/0 is undefined.) The reciprocal also is called the *multiplicative inverse*, since $a \times 1/a = 1$. For example, the reciprocal of 8 is $\frac{1}{8}$; the reciprocal of $\frac{4}{7}$ is $\frac{7}{4}$.

Least Common Denominator: Fractions cannot be added or subtracted without a common denominator. For example, $\frac{2}{5} + \frac{1}{5} = \frac{2+1}{5} = \frac{3}{5}$, a simple computation, since the denominator in the answer is the same denominator seen in the fractions. In general, $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$. But fractions with different denominators cannot be added or subtracted until they are converted to *equivalent forms* that have common denominators. This is done by raising the fractions to higher terms (as explained previously). While any common multiple serves as a common denominator, it is preferable to use the *least common multiple* (LCM) of the denominator, referred to as the *least common denominator* (LCD). For example, 18 is the LCD of $\frac{2}{9}$ and $\frac{5}{6}$, since the LCM of 9 and 6 is 18. Raising each fraction to its equivalent form having a denominator of 18 is shown:

$$\frac{2}{9} \times \frac{2}{2} = \frac{4}{18} \quad \text{and} \quad \frac{5}{6} \times \frac{3}{3} = \frac{15}{18}$$

Example: In the case of $\frac{9}{11}$ and $\frac{7}{10}$ the LCD is the product of the denominators, $11 \times 10 = 110$. Raising each fraction to its equivalent form is shown:

$$\frac{9}{11} \times \frac{10}{10} = \frac{90}{110} \quad \text{and} \quad \frac{7}{10} \times \frac{11}{11} = \frac{77}{110}$$

Adding and Subtracting Fractions and Mixed Numbers

To Add or Subtract Common Fractions: 1) Convert each fraction to terms of the least common denominator; 2) add or subtract numerators; 3) if answer is an improper fraction, change it to a mixed number; and 4) reduce fraction part if necessary.

Example, Addition of Common Fractions

$$\begin{array}{lcccc} \frac{1}{4} & \xrightarrow{\text{LCD} = 16} & \frac{1}{4}\times\frac{4}{4} & = & \frac{4}{16} \\ \frac{3}{16} & & \frac{3}{16}\times\frac{1}{1} & = & \frac{3}{16} \\ +\frac{7}{8} & & +\frac{7}{8}\times\frac{2}{2} & = & +\frac{14}{16} \\ \hline & & & & \frac{21}{16} \end{array}$$

Example, Subtraction of Common Fractions

$$\begin{array}{lcccc} \frac{15}{16} & \xrightarrow{\text{LCD} = 48} & \frac{15}{16}\times\frac{3}{3} & = & \frac{45}{48} \\ -\frac{7}{12} & & -\frac{7}{12}\times\frac{4}{4} & = & -\frac{28}{48} \\ \hline & & & & \frac{17}{48} \end{array}$$

To Add Mixed Numbers: Two methods for adding mixed numbers are shown below the explanations.

First method: 1) Raise fraction parts to the higher terms of the LCD; 2) add whole number parts and fraction parts separately; 3) if result has an improper fraction, convert it to a mixed number and add the whole number parts.

Second method: 1) Convert mixed numbers to improper fractions; 2) raise resulting fractions to the higher terms of the LCD; 3) add fractions as usual and convert back to a mixed number; reduce, if needed.

Examples, Addition of Mixed Numbers

Method 1

$$\begin{array}{lcccc} 2\frac{1}{2} & & 2\frac{1}{2}\times\frac{16}{16} & = & 2\frac{16}{32} \\ 4\frac{1}{4} & \rightarrow & 4\frac{1}{4}\times\frac{8}{8} & = & 4\frac{8}{32} \\ +1\frac{15}{32} & & +1\frac{15}{32}\times\frac{1}{1} & = & +1\frac{15}{32} \\ \hline & & & & 7\frac{39}{32} = 7 + 1\frac{7}{32} = 8\frac{7}{32} \end{array}$$

Method 2

$$\begin{array}{lccccc} 2\frac{1}{2} & & \frac{5}{2} = \frac{5}{2}\times\frac{16}{16} & = & \frac{80}{32} \\ 4\frac{1}{4} & \rightarrow & \frac{17}{4} = \frac{17}{4}\times\frac{8}{8} & = & \frac{136}{32} \\ +1\frac{15}{32} & & +\frac{47}{32} = \frac{47}{32}\times\frac{1}{1} & = & +\frac{47}{32} \\ \hline & & & & \frac{263}{32} = 8\frac{7}{32} \end{array}$$

To Subtract Mixed Numbers: The methods are similar to those for adding, except the fraction part may need to "borrow" from the whole number. The examples show the details.

1) Convert fraction parts to equivalent fractions with LCD; 2) subtract whole number and fraction parts separately, *unless* the first fraction's numerator is *smaller* than the second. In that case, proceed as shown in the second and third examples below, borrowing 1 in the form of a fraction and then subtracting.

Examples, Subtraction of Mixed Numbers

Example 1

$$\begin{array}{r} 12\frac{4}{5} \\ -4\frac{1}{5} \\ \hline 8\frac{3}{5} \end{array}$$

Example 2

$$\begin{array}{rcl} 43\frac{14}{15} & = & 43\frac{14}{15} \\ -19\frac{3}{5} & = & -19\frac{9}{15} \\ \hline & & 24\frac{5}{15} = 24\frac{1}{3} \end{array}$$

Example 3

$$\begin{array}{rcl} 20 = 19 + 1 & = & 19\frac{2}{2} \\ -7\frac{1}{2} = -7\frac{1}{2} & = & -7\frac{1}{2} \\ \hline & & 12\frac{1}{2} \end{array}$$

Example 4

$$\begin{array}{rcl} 8\frac{2}{9} = 7 + 1\frac{2}{9} & = & 7\frac{11}{9} \\ -1\frac{4}{9} = -1\frac{4}{9} & = & -1\frac{4}{9} \\ \hline & & 6\frac{7}{9} \end{array}$$

Multiplying Fractions and Mixed Numbers

To Multiply Common Fractions: 1) Multiply numerators; 2) multiply denominators; and 3) convert improper fractions to mixed numbers, if necessary.

To Multiply Mixed Numbers: 1) Convert mixed numbers to improper fractions; 2) multiply numerators; 3) multiply denominators; and 4) convert improper fractions to mixed numbers, if necessary.

Examples, Multiplication of Fractions

$$\frac{2}{3}\times\frac{8}{15} = \frac{2\times8}{3\times15} = \frac{16}{45} \qquad 4\frac{2}{5}\times1\frac{1}{3} = \frac{22}{5}\times\frac{4}{3} = \frac{22\times4}{5\times3} = \frac{88}{15}$$

Dividing Fractions and Mixed Numbers

To Divide Common Fractions: 1) Take the reciprocal of the dividing fraction; 2) multiply the numerators and denominators; and 3) convert improper fractions to mixed numbers, if necessary.

To Divide Mixed Numbers: 1) Convert the mixed numbers to improper fractions; 2) take the reciprocal of the dividing fraction; 3) multiply numerators and denominators; and 4) convert improper fractions to mixed numbers, if necessary.

Examples, Division of Fractions

$$\frac{2}{7} \div \frac{5}{21} = \frac{2}{7} \times \frac{21}{5} = \frac{2 \times 21}{7 \times 5} = \frac{42}{35} = \frac{6}{5} \qquad 3\frac{1}{3} \div 2\frac{4}{5} = \frac{10}{3} \div \frac{14}{5} = \frac{10}{3} \times \frac{5}{14} = \frac{50}{42} = \frac{25}{21}$$

Decimal Numbers.—Decimal fractions are fractional parts of a whole whose implied denominators are multiples of 10. A decimal fraction of 0.1 has a value of 1/10, 0.01 has a value of 1/100, 0.001 has a value of 1/1000, and so on. Thus, the value of the digit in the first place right of the decimal point is expressed in tenths, a digit two places to the right is expressed in hundredths, a digit three places to the right is expressed in thousandths, and so on. Because the denominator is implied, the number to the right of the decimal point indicates the numerator of the decimal fraction. For example, 0.125 is equivalent to 125/1000.

In industry, most decimal fractions are expressed in terms of thousandths rather than tenths or hundredths. For example, a decimal fraction of 0.2 is written as 0.200 and read as "200 thousandths" rather than "2 tenths"; a value of 0.75 is written as 0.750, and read as "750 thousandths" rather than "75 hundredths." In the case of four place decimals, the values are expressed in terms of ten-thousandths. So a value of 0.1875 is read as "1875 ten-thousandths."

Just as a mixed number is the sum of a whole number and a fraction, a decimal number greater than 1 has a whole and a decimal part. For example, $10.125 = 10^{125}/_{1000}$, which is read as "10 and 125 thousandths."

Adding or Subtracting Decimal Numbers: To add or subtract decimal numbers, align the decimal points and add or subtract the digits as usual. The decimal point in the answer is aligned with the decimal points in the numbers added or subtracted.

Examples, Adding Decimal Fractions

0.125	1.750
1.0625	0.875
2.50	0.125
+ 0.1875	+ 2.0005
3.8750	4.7505

Examples, Subtracting Decimal Fractions

1.750	2.625
−0.250	−1.125
1.500	1.500

Multiplying Decimal Numbers: In setting up decimal multiplication, the decimal points do not have to be aligned. Long multiplication is done as usual, but the decimal point in the answer is placed so that the number of digits on its right is the same as the total number of digits on the right of the numbers multiplied.

Examples, Decimal Number Multiplication

24.035	three decimal places
× 0.08	two decimal places
1.92280	five decimal places

6.002	three decimal places
× 41.3	one decimal place
18006	
60020	
+ 2400800	
247.8826	four decimal places

Dividing Decimal Numbers: There are several types of decimal division problems: (1) a whole number divided by a decimal number; (2) a decimal number divided by a whole number; and (3) a decimal number divided by a decimal number. For all situations, if the divisor is a decimal, its decimal point must first be moved right to make it a whole number, and the dividend's decimal likewise moved, before the operation is performed. Examples of each type are: $18 \div 0.3 = 180 \div 3 = 60$; $1.8 \div 3 = 0.6$; $1.8 \div 0.003 = 1800 \div 3 = 600$.

Ratio and Proportion.—A *ratio* of quantities a to b is written $a{:}b$ or as a fraction a/b. For example, the ratio of 12 to 3 is written 12:3 or 12/3. Ratios, like fractions, can be reduced: 12:3 is 4:1. The *inverse* (or reciprocal) ratio of $a{:}b$ is $b{:}a$. Thus, the inverse ratio of 12:3 is 3:12.

When two or more ratios are multiplied, the ratio obtained is a called a *compound ratio*. The compound ratio of $a{:}b$, $c{:}d$, and $e{:}f$ is the ratio $ace{:}bdf$. For example, the compound ratio of 8:2, 9:3, and 10:5 is $8 \times 9 \times 10{:}\ 2 \times 3 \times 5$, or 720:30.

An equality of ratios, $a/b = c/d$, is called a *proportion*, which can be written as $a{:}b{::}c{:}d$, read as "a is to b as c is to d." Thus, 6:3::10:5 because 6/3 and 10/5 both reduce to 2/1 or 2. In a proportion $a{:}b{::}c{:}d$, the first and last terms (which can be variables or numbers) are called the *extremes*, and the second and third are the *means*.

Note that if $a/b = c/d$, then the rules of algebra show that $ad = bc$. Thus, the proportion $a{:}b{::}c{:}d$ is equivalent to $a \times d = b \times c$. So the proportion 6:3::10:5 is equivalent to $6 \times 5 = 3 \times 10$.

Often, some part of a proportion is an unknown. For example, in the proportion $2{:}3{::}n{:}4$ (2 is to 3 as n is to 4), n is found by setting up a proportion. According to the basic rules of algebra, $2{:}3{::}n{:}4$ means $(2)(4) = 3n$, and hence, $8 = 3n$, so $n = 8/3$. A full discussion of the rules for solving equations can be found in *ALGEBRA*.

If the second and third terms in a proportion are the same, that term is the *mean proportional* of the other two. Thus, in the proportion 8:4::4:2, 4 is the mean proportional of 8 and 2. The mean proportional of any two numbers may be found by multiplying them and extracting the square root of the product. Thus, the mean proportional of 3 and 12 is 6, because $3 \times 12 = 36$, which is 6^2.

Example 1, Involving Proportion: If it takes 18 days to assemble 4 lathes, how many days would it take to assemble 14 lathes?

Solution: Let x be the number of days to be found. The proportion is written $4{:}18 :: 14{:}x$, where x is the number of days to be found. Setting this up as an equation and solving:

$$\frac{4}{18} = \frac{14}{x}$$

$$x = \frac{18 \times 14}{4} = 63 \text{ days}$$

Example 2, Involving Direct (Simple) Proportion: 10 linear meters (32.81 feet) of bar stock are required as blanks for 100 clamping bolts. What total length x of stock, in meters and feet, is required for 912 bolts?

Solution: The setup to solve the proportional meters-to-bolts problem comes from the way this proportion is read: "10 meters is to 100 bolts as how many meters is to 912 bolts." It is solved accordingly:

$$10 : 100 :: x : 912, \text{ that is, } \frac{10}{100} = \frac{x}{912} \qquad \text{Solving for } x\text{: } x = \frac{10 \times 912}{100} = \frac{9120}{100} = 91.2 \text{ meters}$$

Likewise, the setup to solve the feet-to-bolts problem comes from reading it as: "32.81 feet is to 100 bolts as how many feet is to 912 bolts." Thus:

$$32.81 : 100 :: x : 912, \text{ that is, } \frac{32.81}{100} = \frac{x}{912} \qquad \text{Solving for } x\text{: } x = \frac{32.81 \times 912}{100} = \frac{29{,}922.72}{100} = 299.2 \text{ feet}$$

Inverse Proportion: Quantities with an inversely proportional relationship behave in such a way that as one increases the other decreases. For example, a factory employing 270 workers completes a given number of automotive components weekly, the number of working hours being 44 per week. If the hours are reduced, then more workers will be required to do the same amount of work. How many employees would be required for the same production if the working hours were reduced to 40 per week?

The hours per week is inversely proportional to the number of workers; fewer hours per worker means more workers are required. Letting x be the number of workers needed when time is reduced, the inverse proportion is written:

$$270 : x \; :: \; 40 : 44$$

Thus

$$\frac{270}{x} = \frac{40}{44} \quad \text{and} \quad x = \frac{270 \times 44}{40} = 297 \text{ workers}$$

Problems Involving Both Direct and Inverse Proportions: If two groups of data are related by both direct (simple) and inverse proportions among the various quantities, a simple mathematical relation may be used to solve the problem as follows:

$$\frac{\text{Product of all directly proportional items in first group}}{\text{Product of all inversely proportional items in first group}}$$

$$= \frac{\text{Product of all directly proportional items in second group}}{\text{Product of all inversely proportional items in second group}}$$

Example: If a worker capable of turning 65 studs in a 10-hour day is paid $13.50 per hour, how much per hour should a worker be paid who turns 72 studs in a 9-hour day if compensated in the same proportion as the first worker?

Solution: The first group of data in this problem consists of the number of hours worked, the hourly wage of the first worker, and the number of studs produced per day; the second group contains similar data for the second worker, except the hourly wage is unknown, so it is indicated by x.

The labor cost per stud, as may be seen, is directly proportional to the number of hours worked and the hourly wage. These quantities, therefore, are used in the numerators of the fractions in the formula. The labor cost per stud is inversely proportional to the number of studs produced per day. (The greater the number of studs produced in a given time the less the cost per stud.) The numbers of studs per day, therefore, are placed in the denominators of the fractions in the formula. Thus,

$$\frac{(10)(13.50)}{65} = \frac{9x}{72}$$

$$x = \frac{(10)(13.50)(72)}{(65)(9)} = \$16.62 \text{ per hour}$$

Percentage.—A percentage is a ratio expressed as a part of 100. For example, if out of 100 manufactured parts, 12 do not pass inspection, then 12 percent (12 of the 100) are rejected. The symbol % indicates percentage.

The percent of gain (or loss) with respect to a base (original) quantity is found by dividing the amount of gain (or loss) by the base quantity and multiplying the quotient by 100. For example, if a quantity of steel is bought for $2000 and sold for $2500, the profit is $500/2000 × 100, or 25 percent of the invested amount.

Example: Out of a total output of 280 castings a day, 30 castings are, on average, rejected. What is the percentage of bad castings?

$$\frac{30}{280} \times 100 = 10.71 \text{ percent}$$

Percent Change: Any increase or decrease in some measured quantity can be expressed as a *percent change* using the formula: $\frac{\text{final} - \text{original amount}}{\text{original}} \times 100 = \text{percent change}$. If in the

previous example, production increased from 280 to 300, then the percent change would be:
$\frac{\text{final} - \text{original amount}}{\text{original}} \times 100 = \frac{300 - 280}{280} \times 100 = \frac{20}{280} \times 100 = 7.14\%$.

The denominator is always the original amount. Percent change also can be negative. If production decreased from 280 to 245, the percent change would be: $\frac{245 - 280}{280} \times 100 = \frac{-35}{280} \times 100 = -12.5\%$.

Powers and Roots

Powers: The *square* or *second power* of a number (or quantity) is the product of that number multiplied by itself. Thus, the square of 9 is 9×9. The square of a number is indicated by the *exponent*^{2}, thus: $9^2 = 9 \times 9 = 81$.

The *cube* or *third power* of a number n is the product $n \times n \times n$, or n^3. Thus, the cube of 4 is $4 \times 4 \times 4 = 64$, and is written 4^3.

In general, the nth power of a is written a^n, where a is the *base* and n is the *exponent*.

Roots: The *square root* of a given number is the positive number which, when multiplied by itself, will produce the given number. The square root of 16 (written $\sqrt{16}$) is 4 because $4 \times 4 = 16$. The other root of 16 is -4, but the use of the square root symbol indicates the positive (principal) square root only.

Similarly, the *cube root* of a given number is the number which, when used as a factor three times, will produce the given number. Thus, the cube root of 64 (written $\sqrt[3]{64}$) is 4 because $4 \times 4 \times 4 = 64$.

In general, the nth root of a is written as $\sqrt[n]{a}$ or $a^{1/n}$.

Properties of Exponents

$a^n a^m = a^{n+m}$	$\frac{a^n}{a^m} = a^{(n-m)}$	$(a^m)^n = a^{mn}$	$(ab)^m = a^m b^m$
$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$a^{m/n} = (a^{1/n})^m$ or $(a^m)^{1/n}$	$a^{-n} = \frac{1}{a^n}$	$\frac{1}{a^{-n}} = a^n$
$a^0 = 1 \; (a \neq 0)$	$a^{1/n} = \sqrt[n]{a}$	$a^{m/n} = (\sqrt[n]{a})^m = \left(\sqrt[n]{a^m}\right)$ unless $a < 0$, and m and n are both even	

Examples:

$3^1 3^2 = 3^{1+2} = 3^3 = 27$	$(x)(x^3) = x^{(1+3)} = x^4$
$\frac{5^4}{5^2} = 5^{4-2} = 5^2 = 25$	$\frac{x^9}{x^6} = x^{(9-6)} = x^3$
$(2^4)^2 = 2^{(4)(2)} = 2^8 = 256$	$(x^3)^3 = x^{(3)(3)} = x^9$
$(9x)^2 = 9^2 x^2 = 81x^2$	$(ab^4)^2 = a^2 b^8$
$32^{3/5} = (32^{1/5})^3 = (\sqrt[5]{32})^3 = 2^3 = 8$	$4^{-3} = \frac{1}{4^3} = \frac{1}{64}$
$\frac{1}{2^{-5}} = 2^5 = 32$	$9x^0 = 9(1) = 9$

Using logarithms can greatly facilitate the process of raising a number to a power or extracting its root. As shown in *Logarithms* on page 31, this is especially true if the power is not an integer. For example, the square root of 137.1 can only be found with a degree of accuracy through logarithms, a scientific calculator, or Taylor series polynomials.

Scientific Notation.—Calculations involving both large and small magnitude numbers are facilitated by *scientific notation*. In this system, a number is expressed by two factors: (1) an integer from 1 to 9, possibly followed by a decimal, and (2) a power of 10. Large numbers in standard form are converted to scientific notation as shown in the following examples:

$$50{,}000 = 5 \times 10^4 \qquad 273.15 = 2.7315 \times 10^2$$

In the example, 50,000 becomes 5×10^4 because the positive exponent on 10 is the number of places to the right that the decimal point moves so that the first factor falls between 1 and 10. Numbers less than 1 are converted to scientific notation as shown in the following examples:

$$0.840 = 8.40 \times 10^{-1} \qquad 0.0000001 = 1 \times 10^{-7}$$

The negative exponent shows the number of places to the left that the decimal point moves, so that the first factor falls between 1 and 10.

Science and engineering quantities—which are often quite large or small—lend themselves to representation in scientific notation. For instance, *Avogadro's number*, which is the number of particles in one mole of a substance, is 6.024×10^{23}. The metric (SI) pressure unit of 1 pascal (Pa) is equivalent to 0.00000986923 atmosphere (atm) or 0.0001450377 pound/square inch (psi). In scientific notation, these figures are 9.86923×10^{-6} atm and 1.450377×10^{-4} psi, respectively.

Engineering notation is a version of scientific notation in which the exponent of 10 is always a multiple of 3. (See *MEASURING UNITS* on page 2836 for a table of this system.)

Multiplication in Scientific Notation: The procedure is as follows:

1) Multiply the first factors of the numbers to obtain the first factor of the product.

2) Add the exponents of the factors of 10 to obtain the product's factor of 10. Thus:

$$(4.31 \times 10^{-2}) \times (9.01 \times 10) = (4.31 \times 9.01) \times 10^{-2+1} = 38.8331 \times 10^{-1}$$

$$(5.98 \times 10^4) \times (4.37 \times 10^3) = (5.98 \times 4.37) \times 10^{4+3} = 26.1326 \times 10^7$$

3) Write the final in conventional scientific notation, as explained in the previous section. So, for the two examples:

$38.8331 \times 10^{-1} = 3.88331 \times 10^0 = 3.88331$, because $10^0 = 1$, and $26.1326 \times 10^7 = 2.61326 \times 10^8$.

When multiplying several numbers written in this notation, the procedure is the same. Thus, $(4.02 \times 10^{-3}) \times (3.987 \times 10) \times (4.863 \times 10^5) = (4.02 \times 3.987 \times 4.863) \times 10^{(-3+1+5)} = 77.94 \times 10^3 = 7.79 \times 10^4$, rounding off the first factor to two decimal places.

Division in Scientific Notation: The procedure is as follows:

1) Divide the first factor of the dividend (the first number) by the first factor of the divisor (the second number) to get the first factor of the quotient.

2) Subtract the exponents of the factors of 10 to obtain the product's factor of 10:

$$(4.31 \times 10^{-2}) \div (9.0125 \times 10) =$$

$$(4.31 \div 9.0125) \times (10^{-2-1}) = 0.4782 \times 10^{-3} = 4.782 \times 10^{-4}$$

It can be seen that this system of notation is helpful where several numbers of different magnitudes are to be multiplied and divided.

Example: Find the solution of $\dfrac{250 \times 4698 \times 0.00039}{43678 \times 0.002 \times 0.0147}$

Solution: Changing all these numbers to powers of 10 notation and performing the operations indicated:

$$\frac{(2.5\times10^{2})\times(4.698\times10^{3})\times(3.9\times10^{-4})}{(4.3678\times10^{4})\times(2\times10^{-3})\times(1.47\times10^{-2})}$$

$$= \frac{(2.5\times4.698\times3.9)(10^{2+3-4})}{(4.3678\times2\times1.47)(10^{4-3-2})} = \frac{45.8055\times10}{12.8413\times10^{-1}}$$

$$= 3.5670\times10^{1-(-1)} = 3.5670\times10^{2} = 356.70 \text{ (rounded)}$$

Factorial Notation.—A factorial is a mathematical shortcut denoted by the symbol **!** following a number (for example, 3! is "three factorial"). $n!$ is found by multiplying together all the positive integers less than or equal to the factorial number n. Zero factorial (0!) is defined as 1. For example: $3! = 1\times2\times3 = 6$; $4! = 1\times2\times3\times4 = 24$; $7! = 1\times2\times3\times4\times5\times6\times7 = 5040$; etc. Factorial notation is used in certain areas, including probability and analysis. The following two topics (permutations and combinations) relate to probability and statistics.

Permutation.—A permutation is an arrangement of objects of a set into a sequence or order. In mathematics, the number of arrangements of n objects is given by $n!$. For example, 4 objects can be arranged 4! ways, that is, $4\times3\times2\times1 = 24$ ways. The number of ways r objects can be arranged (that is, ordered) from a set of n is given by the *permutation* formula $_nP_r = \frac{n!}{(n-r)!}$

Example: How many ways can the letters X, Y, and Z be arranged?

Solution: Three objects ($r = 3$) out of a set of 3 ($n = 3$) are being arranged. The numbers of possible arrangements for the three letters are $3!/(3-3)! = (3\times2\times1)/1 = 6$. Listing them is not difficult, since there are so few: XYZ, XZY, YXZ, YZX, ZXY, ZYX.

Example: There are 10 people participating in a foot race. How many arrangements of first, second, and third place winners are there?

Solution: Here r is 3 and n is 10. The number of possible arrangements of winners are:

$$_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 10\times9\times8 = 720$$

Combination.—This is the number of ways r objects can be chosen from n in a way that order does not matter. It is expressed as "n choose r." There are fewer combinations than permutations of r objects out of n, since it does not matter in what order the three objects are chosen. So in a combination, choosing ABC is the same as choosing ACB or BAC and so on. The formula is $_nC_r = \frac{n!}{(n-r)!r!}$

Example: How many possible sets of 6 numbers can be picked with no regard for order from the numbers 1 to 52?

Solution: Here r is 6 and n is 52. So the possible number of combinations is:

$$_{52}C_6 = \frac{52!}{(52-6)!6!} = \frac{52!}{46!6!} = \frac{52\times51\times50\times49\times48\times47}{1\times2\times3\times4\times5\times6} = 20{,}358{,}520$$

Prime Factorization of Numbers.—Tables of prime numbers and factors of numbers are particularly useful for calculations involving change-gear ratios for compound gearing, dividing heads, gear-generating machines, and mechanical designs having gear trains.

Definition: p is a *factor* of a number n if the division n/p leaves no remainder. Thus, any number n has factors of itself and 1, because $n/n = 1$ and $n/1 = n$. Other factors of a number are found as follows:

2 is a factor of any even number. Thus, $28 = 2\times14$, and $210 = 2\times105$.

3 is a factor of any number where the sum of its digits is divisible by 3. Thus, 3 is a factor of 2397, because $2+3+9+7 = 21$, and $21\div3 = 7$.

4 is a factor of any number in which the last two digits are a number divisible by 4. Thus, 1844 has a factor 4, because 44 ÷ 4 = 11. 761 does not have a factor of 4, since 61 is not divisible by 4.

5 is a factor of any number that has a ones digit that is either 0 or 5.

A *prime number* is one that has no factors except itself and 1. Thus, 2, 3, 5, 7, 11, etc., are prime numbers. 2 is the only even prime number. A factor which itself is a prime number is called a *prime factor*. All numbers can be expressed as a product of their prime factors.

It can be determined if 7 is a factor of a number according to this process: Remove the last digit from the number, double it, and subtract it from the remaining number. If the result is divisible by 7 (e.g., 14, 7, 0, –7, etc.), then 7 is a prime factor of the number.

The *prime factorization* of a number is done by expressing the number as a product of its primes. For example, the prime factors of 20 are 2 and 5. The prime factorization is 2 × 2 × 5 = 20.

Table 1 gives the smallest prime factor of all odd numbers from 1 to 2399. This table can be used for finding all the factors for numbers up to this odd number. Where no factor is given for a number in the table, the letter **P** indicates that the number is a prime number. *Table 2* lists prime numbers from 2 through 15277 and can be used to identify unfactorable numbers in that range. (More prime number tables are included in the *ADDITIONAL MATERIAL* in the *Machinery's Handbook 32 Digital Edition*.)

Example 1: Find the factors of 833. Use the table on page 15 as illustrated below.

Solution: The table on page 15 indicates that 7 is the smallest prime factor of 833, shown at the row-column intersection for 833. This leaves another factor, because 833 ÷ 7 = 119.

From To	0 100	100 200	200 300	300 400	400 500	500 600	600 700	700 800	800 900	900 1000	1000 1100	1100 1200
…	…	…	…	…	…	…	…	…	…	…	…	…
33	3	7	P	3	P	13	3	P	**7**	3	P	11
…	…	…	…	…	…	…	…	…	…	…	…	…

Looking up 119, it also shows that 7 is a prime factor of 119, leaving a factor 119 ÷ 7 = 17.

From To	0 100	100 200	200 300	300 400	400 500	500 600	600 700	700 800	800 900	900 1000	1000 1100	1100 1200
…	…	…	…	…	…	…	…	…	…	…	…	…
19	P	7	3	11	P	3	P	P	3	P	P	3
…	…	…	…	…	…	…	…	…	…	…	…	…

Looking up 17, **P** indicates that 17 is a prime number and no other prime factors of 833 exist.

From To	0 100	100 200	200 300	300 400	400 500	500 600	600 700	700 800	800 900	900 1000	1000 1100	1100 1200
…	…	…	…	…	…	…	…	…	…	…	…	…
17	**P**	3	7	P	3	11	P	3	19	7	3	P
…	…	…	…	…	…	…	…	…	…	…	…	…

Hence, the prime factorization of 833 is 7 × 7 × 17.

Example 2: A set of four gears is required in a mechanical design to provide an overall gear ratio of 3332 ÷ 1200. Furthermore, no gear in the set is to have more than 120 teeth or less than 24 teeth. Determine the tooth numbers.

Solution: The prime factorization of 3332 is determined to be 2 × 2 × 7 × 7 × 17 = 3332. The prime factorization of 1200 is 2 × 2 × 2 × 2 × 5 × 5 × 3 = 1200. Therefore,

$$\frac{3332}{1200} = \frac{2 \times 2 \times 7 \times 7 \times 17}{2 \times 2 \times 2 \times 2 \times 5 \times 5 \times 3} = \left(\frac{68}{48}\right)\left(\frac{49}{25}\right)$$

Each resulting factor represents the number of teeth that fulfill the requirement. Note that the first gear set having 68 and 48 teeth can be reduced in size to gears having 34 and 24 teeth, or even 17 and 12 teeth, while preserving the same gear ratio, since both 3332 and 1200 are divisible by 4.

Example 3: Factor 21,194 into two numbers, neither of which is larger than 200.

Solution: The smallest factor of 21,194 is obviously 2, leaving 21,194 ÷ 2 = 10,597 to be factored further. However, in Table 2a, starting on page 17, which includes the prime numbers from 2 to 15,277, it can be seen that 10,597 is a prime number; therefore, no other factors exist. So the factorization named is not possible.

An approximate factorization with two numbers can be quickly found by taking the square root of 21,194, which equals 145.582. The two factors 145 and 146 provide a product of 21,170, which has an error of 0.113 percent, relative to the desired number 21,194.

Table 1a. Prime Number and Factor Table for 1 to 1199

From To	0 100	100 200	200 300	300 400	400 500	500 600	600 700	700 800	800 900	900 1000	1000 1100	1100 1200
1	**P**	**P**	3	7	**P**	3	**P**	**P**	3	17	7	3
2	**P**	2	2	2	2	2	2	2	2	2	2	2
3	**P**	**P**	7	3	13	**P**	3	19	11	3	17	**P**
5	**P**	3	5	5	3	5	5	3	5	5	3	5
7	**P**	**P**	3	**P**	11	3	**P**	7	3	**P**	19	3
9	3	**P**	11	3	**P**	**P**	3	**P**	**P**	3	**P**	**P**
11	**P**	3	**P**	**P**	3	7	13	3	**P**	**P**	3	11
13	**P**	**P**	3	**P**	7	3	**P**	23	3	11	**P**	3
15	3	5	5	3	5	5	3	5	5	3	5	5
17	**P**	3	7	**P**	3	11	**P**	3	19	7	3	**P**
19	**P**	7	3	11	**P**	3	**P**	**P**	3	**P**	**P**	3
21	3	11	13	3	**P**	**P**	3	7	**P**	3	**P**	19
23	**P**	3	**P**	17	3	**P**	7	3	**P**	13	3	**P**
25	5	5	3	5	5	3	5	5	3	5	5	3
27	3	**P**	**P**	3	7	17	3	**P**	**P**	3	13	7
29	**P**	3	**P**	7	3	23	17	3	**P**	**P**	3	**P**
31	**P**	**P**	3	**P**	**P**	3	**P**	17	3	7	**P**	3
33	3	7	**P**	3	**P**	13	3	**P**	7	3	**P**	11
35	5	3	5	5	3	5	5	3	5	5	3	5
37	**P**	**P**	3	**P**	19	3	7	11	3	**P**	17	3
39	3	**P**	**P**	3	**P**	7	3	**P**	**P**	3	**P**	17
41	**P**	3	**P**	11	3	**P**	**P**	3	29	**P**	3	7
43	**P**	11	3	7	**P**	3	**P**	**P**	3	23	7	3
45	3	5	5	3	5	5	3	5	5	3	5	5
47	**P**	3	13	**P**	3	**P**	**P**	3	7	**P**	3	31
49	7	**P**	3	**P**	**P**	3	11	7	3	13	**P**	3
51	3	**P**	**P**	3	11	19	3	**P**	23	3	**P**	**P**
53	**P**	3	11	**P**	3	7	**P**	3	**P**	**P**	3	**P**
55	5	5	3	5	5	3	5	5	3	5	5	3
57	3	**P**	**P**	3	**P**	**P**	3	**P**	**P**	3	7	13
59	**P**	3	7	**P**	3	13	**P**	3	**P**	7	3	19
61	**P**	7	3	19	**P**	3	**P**	**P**	3	31	**P**	3
63	3	**P**	**P**	3	**P**	**P**	3	7	**P**	3	**P**	**P**
65	5	3	5	5	3	5	5	3	5	5	3	5
67	**P**	**P**	3	**P**	**P**	3	23	13	3	**P**	11	3
69	3	13	**P**	3	7	**P**	3	**P**	11	3	**P**	7
71	**P**	3	**P**	7	3	**P**	11	3	13	**P**	3	**P**
73	**P**	**P**	3	**P**	11	3	**P**	**P**	3	7	29	3
75	3	5	5	3	5	5	3	5	5	3	5	5
77	7	3	**P**	13	3	**P**	**P**	3	**P**	**P**	3	11
79	**P**	**P**	3	**P**	**P**	3	7	19	3	11	13	3
81	3	**P**	**P**	3	13	7	3	11	**P**	3	23	**P**
83	**P**	3	**P**	**P**	3	11	**P**	3	**P**	**P**	3	7
85	5	5	3	5	5	3	5	5	3	5	5	3
87	3	11	7	3	**P**	**P**	3	**P**	**P**	3	**P**	**P**
89	**P**	3	17	**P**	3	19	13	3	7	23	3	29
91	7	**P**	3	17	**P**	3	**P**	7	3	**P**	**P**	3
93	3	**P**	**P**	3	17	**P**	3	13	19	3	**P**	**P**
95	5	3	5	5	3	5	5	3	5	5	3	5
97	**P**	**P**	3	**P**	7	3	17	**P**	3	**P**	**P**	3
99	3	**P**	13	3	**P**	**P**	3	17	29	3	7	11

Table 1b. Prime Number and Factor Table for 1201 to 2399

From To	1200 1300	1300 1400	1400 1500	1500 1600	1600 1700	1700 1800	1800 1900	1900 2000	2000 2100	2100 2200	2200 2300	2300 2400
1	**P**	**P**	3	19	**P**	3	**P**	**P**	3	11	31	3
3	3	**P**	23	3	7	13	3	11	**P**	3	**P**	7
5	5	3	5	5	3	5	5	3	5	5	3	5
7	17	**P**	3	11	**P**	3	13	**P**	3	7	**P**	3
9	3	7	**P**	3	**P**	**P**	3	23	7	3	47	**P**
11	7	3	17	**P**	3	29	**P**	3	**P**	**P**	3	**P**
13	**P**	13	3	17	**P**	3	7	**P**	3	**P**	**P**	3
15	3	5	5	3	5	5	3	5	5	3	5	5
17	**P**	3	13	37	3	17	23	3	**P**	29	3	7
19	23	**P**	3	7	**P**	3	17	19	3	13	7	3
21	3	**P**	7	3	**P**	**P**	3	17	43	3	**P**	11
23	**P**	3	**P**	**P**	3	**P**	**P**	3	7	11	3	23
25	5	5	3	5	5	3	5	5	3	5	5	3
27	3	**P**	**P**	3	**P**	11	3	41	**P**	3	17	13
29	**P**	3	**P**	11	3	7	31	3	**P**	**P**	3	17
31	**P**	11	3	**P**	7	3	**P**	**P**	3	**P**	23	3
33	3	31	**P**	3	23	**P**	3	**P**	19	3	7	**P**
35	5	3	5	5	3	5	5	3	5	5	3	5
37	**P**	7	3	29	**P**	3	11	13	3	**P**	**P**	3
39	3	13	**P**	3	11	37	3	7	**P**	3	**P**	**P**
41	17	3	11	23	3	**P**	7	3	13	**P**	3	**P**
43	11	17	3	**P**	31	3	19	29	3	**P**	**P**	3
45	3	5	5	3	5	5	3	5	5	3	5	5
47	29	3	**P**	7	3	**P**	**P**	3	23	19	3	**P**
49	**P**	19	3	**P**	17	3	43	**P**	3	7	13	3
51	3	7	**P**	3	13	17	3	**P**	7	3	**P**	**P**
53	7	3	**P**	**P**	3	**P**	17	3	**P**	**P**	3	13
55	5	5	3	5	5	3	5	5	3	5	5	3
57	3	23	31	3	**P**	7	3	19	11	3	37	**P**
59	**P**	3	**P**	**P**	3	**P**	11	3	29	17	3	7
61	13	**P**	3	7	11	3	**P**	37	3	**P**	7	3
63	3	29	7	3	**P**	41	3	13	**P**	3	31	17
65	5	3	5	5	3	5	5	3	5	5	3	5
67	7	**P**	3	**P**	**P**	3	**P**	7	3	11	**P**	3
69	3	37	13	3	**P**	29	3	11	**P**	3	**P**	23
71	31	3	**P**	**P**	3	7	**P**	3	19	13	3	**P**
73	19	**P**	3	11	7	3	**P**	**P**	3	41	**P**	3
75	3	5	5	3	5	5	3	5	5	3	5	5
77	**P**	3	7	19	3	**P**	**P**	3	31	7	3	**P**
79	**P**	7	3	**P**	23	3	**P**	**P**	3	**P**	43	3
81	3	**P**	**P**	3	41	13	3	7	**P**	3	**P**	**P**
83	**P**	3	**P**	**P**	3	**P**	7	3	**P**	37	3	**P**
85	5	5	3	5	5	3	5	5	3	5	5	3
87	3	19	**P**	3	7	**P**	3	**P**	**P**	3	**P**	7
89	**P**	3	**P**	7	3	**P**	**P**	3	**P**	11	3	**P**
91	**P**	13	3	37	19	3	31	11	3	7	29	3
93	3	7	**P**	3	**P**	11	3	**P**	7	3	**P**	**P**
95	5	3	5	5	3	5	5	3	5	5	3	5
97	**P**	11	3	**P**	**P**	3	7	**P**	3	13	**P**	3
99	3	**P**	**P**	3	**P**	7	3	**P**	**P**	3	11	**P**

Table 2a. Prime Numbers from 2 to 6869

2	347	773	1259	1753	2293	2803	3389	3929	4517	5099	5689	6277
3	349	787	1277	1759	2297	2819	3391	3931	4519	5101	5693	6287
5	353	797	1279	1777	2309	2833	3407	3943	4523	5107	5701	6299
7	359	809	1283	1783	2311	2837	3413	3947	4547	5113	5711	6301
11	367	811	1289	1787	2333	2843	3433	3967	4549	5119	5717	6311
13	373	821	1291	1789	2339	2851	3449	3989	4561	5147	5737	6317
17	379	823	1297	1801	2341	2857	3457	4001	4567	5153	5741	6323
19	383	827	1301	1811	2347	2861	3461	4003	4583	5167	5743	6329
23	389	829	1303	1823	2351	2879	3463	4007	4591	5171	5749	6337
29	397	839	1307	1831	2357	2887	3467	4013	4597	5179	5779	6343
31	401	853	1319	1847	2371	2897	3469	4019	4603	5189	5783	6353
37	409	857	1321	1861	2377	2903	3491	4021	4621	5197	5791	6359
41	419	859	1327	1867	2381	2909	3499	4027	4637	5209	5801	6361
43	421	863	1361	1871	2383	2917	3511	4049	4639	5227	5807	6367
47	431	877	1367	1873	2389	2927	3517	4051	4643	5231	5813	6373
53	433	881	1373	1877	2393	2939	3527	4057	4649	5233	5821	6379
59	439	883	1381	1879	2399	2953	3529	4073	4651	5237	5827	6389
61	443	887	1399	1889	2411	2957	3533	4079	4657	5261	5839	6397
67	449	907	1409	1901	2417	2963	3539	4091	4663	5273	5843	6421
71	457	911	1423	1907	2423	2969	3541	4093	4673	5279	5849	6427
73	461	919	1427	1913	2437	2971	3547	4099	4679	5281	5851	6449
79	463	929	1429	1931	2441	2999	3557	4111	4691	5297	5857	6451
83	467	937	1433	1933	2447	3001	3559	4127	4703	5303	5861	6469
89	479	941	1439	1949	2459	3011	3571	4129	4721	5309	5867	6473
97	487	947	1447	1951	2467	3019	3581	4133	4723	5323	5869	6481
101	491	953	1451	1973	2473	3023	3583	4139	4729	5333	5879	6491
103	499	967	1453	1979	2477	3037	3593	4153	4733	5347	5881	6521
107	503	971	1459	1987	2503	3041	3607	4157	4751	5351	5897	6529
109	509	977	1471	1993	2521	3049	3613	4159	4759	5381	5903	6547
113	521	983	1481	1997	2531	3061	3617	4177	4783	5387	5923	6551
127	523	991	1483	1999	2539	3067	3623	4201	4787	5393	5927	6553
131	541	997	1487	2003	2543	3079	3631	4211	4789	5399	5939	6563
137	547	1009	1489	2011	2549	3083	3637	4217	4793	5407	5953	6569
139	557	1013	1493	2017	2551	3089	3643	4219	4799	5413	5981	6571
149	563	1019	1499	2027	2557	3109	3659	4229	4801	5417	5987	6577
151	569	1021	1511	2029	2579	3119	3671	4231	4813	5419	6007	6581
157	571	1031	1523	2039	2591	3121	3673	4241	4817	5431	6011	6599
163	577	1033	1531	2053	2593	3137	3677	4243	4831	5437	6029	6607
167	587	1039	1543	2063	2609	3163	3691	4253	4861	5441	6037	6619
173	593	1049	1549	2069	2617	3167	3697	4259	4871	5443	6043	6637
179	599	1051	1553	2081	2621	3169	3701	4261	4877	5449	6047	6653
181	601	1061	1559	2083	2633	3181	3709	4271	4889	5471	6053	6659
191	607	1063	1567	2087	2647	3187	3719	4273	4903	5477	6067	6661
193	613	1069	1571	2089	2657	3191	3727	4283	4909	5479	6073	6673
197	617	1087	1579	2099	2659	3203	3733	4289	4919	5483	6079	6679
199	619	1091	1583	2111	2663	3209	3739	4297	4931	5501	6089	6689
211	631	1093	1597	2113	2671	3217	3761	4327	4933	5503	6091	6691
223	641	1097	1601	2129	2677	3221	3767	4337	4937	5507	6101	6701
227	643	1103	1607	2131	2683	3229	3769	4339	4943	5519	6113	6703
229	647	1109	1609	2137	2687	3251	3779	4349	4951	5521	6121	6709
233	653	1117	1613	2141	2689	3253	3793	4357	4957	5527	6131	6719
239	659	1123	1619	2143	2693	3257	3797	4363	4967	5531	6133	6733
241	661	1129	1621	2153	2699	3259	3803	4373	4969	5557	6143	6737
251	673	1151	1627	2161	2707	3271	3821	4391	4973	5563	6151	6761
257	677	1153	1637	2179	2711	3299	3823	4397	4987	5569	6163	6763
263	683	1163	1657	2203	2713	3301	3833	4409	4993	5573	6173	6779
269	691	1171	1663	2207	2719	3307	3847	4421	4999	5581	6197	6781
271	701	1181	1667	2213	2729	3313	3851	4423	5003	5591	6199	6791
277	709	1187	1669	2221	2731	3319	3853	4441	5009	5623	6203	6793
281	719	1193	1693	2237	2741	3323	3863	4447	5011	5639	6211	6803
283	727	1201	1697	2239	2749	3329	3877	4451	5021	5641	6217	6823
293	733	1213	1699	2243	2753	3331	3881	4457	5023	5647	6221	6827
307	739	1217	1709	2251	2767	3343	3889	4463	5039	5651	6229	6829
311	743	1223	1721	2267	2777	3347	3907	4481	5051	5653	6247	6833
313	751	1229	1723	2269	2789	3359	3911	4483	5059	5657	6257	6841
317	757	1231	1733	2273	2791	3361	3917	4493	5077	5659	6263	6857
331	761	1237	1741	2281	2797	3371	3919	4507	5081	5669	6269	6863
337	769	1249	1747	2287	2801	3373	3923	4513	5087	5683	6271	6869

Table 2b. Prime Numbers from 6871 to 15277

6871	7523	8117	8737	9343	9931	10601	11257	11927	12539	13159	13807	14533
6883	7529	8123	8741	9349	9941	10607	11261	11933	12541	13163	13829	14537
6899	7537	8147	8747	9371	9949	10613	11273	11939	12547	13171	13831	14543
6907	7541	8161	8753	9377	9967	10627	11279	11941	12553	13177	13841	14549
6911	7547	8167	8761	9391	9973	10631	11287	11953	12569	13183	13859	14551
6917	7549	8171	8779	9397	10007	10639	11299	11959	12577	13187	13873	14557
6947	7559	8179	8783	9403	10009	10651	11311	11969	12583	13217	13877	14561
6949	7561	8191	8803	9413	10037	10657	11317	11971	12589	13219	13879	14563
6959	7573	8209	8807	9419	10039	10663	11321	11981	12601	13229	13883	14591
6961	7577	8219	8819	9421	10061	10667	11329	11987	12611	13241	13901	14593
6967	7583	8221	8821	9431	10067	10687	11351	12007	12613	13249	13903	14621
6971	7589	8231	8831	9433	10069	10691	11353	12011	12619	13259	13907	14627
6977	7591	8233	8837	9437	10079	10709	11369	12037	12637	13267	13913	14629
6983	7603	8237	8839	9439	10091	10711	11383	12041	12641	13291	13921	14633
6991	7607	8243	8849	9461	10093	10723	11393	12043	12647	13297	13931	14639
6997	7621	8263	8861	9463	10099	10729	11399	12049	12653	13309	13933	14653
7001	7639	8269	8863	9467	10103	10733	11411	12071	12659	13313	13963	14657
7013	7643	8273	8867	9473	10111	10739	11423	12073	12671	13327	13967	14669
7019	7649	8287	8887	9479	10133	10753	11437	12097	12689	13331	13997	14683
7027	7669	8291	8893	9491	10139	10771	11443	12101	12697	13337	13999	14699
7039	7673	8293	8923	9497	10141	10781	11447	12107	12703	13339	14009	14713
7043	7681	8297	8929	9511	10151	10789	11467	12109	12713	13367	14011	14717
7057	7687	8311	8933	9521	10159	10799	11471	12113	12721	13381	14029	14723
7069	7691	8317	8941	9533	10163	10831	11483	12119	12739	13397	14033	14731
7079	7699	8329	8951	9539	10169	10837	11489	12143	12743	13399	14051	14737
7103	7703	8353	8963	9547	10177	10847	11491	12149	12757	13411	14057	14741
7109	7717	8363	8969	9551	10181	10853	11497	12157	12763	13417	14071	14747
7121	7723	8369	8971	9587	10193	10859	11503	12161	12781	13421	14081	14753
7127	7727	8377	8999	9601	10211	10861	11519	12163	12791	13441	14083	14759
7129	7741	8387	9001	9613	10223	10867	11527	12197	12799	13451	14087	14767
7151	7753	8389	9007	9619	10243	10883	11549	12203	12809	13457	14107	14771
7159	7757	8419	9011	9623	10247	10889	11551	12211	12821	13463	14143	14779
7177	7759	8423	9013	9629	10253	10891	11579	12227	12823	13469	14149	14783
7187	7789	8429	9029	9631	10259	10903	11587	12239	12829	13477	14153	14797
7193	7793	8431	9041	9643	10267	10909	11593	12241	12841	13487	14159	14813
7207	7817	8443	9043	9649	10271	10937	11597	12251	12853	13499	14173	14821
7211	7823	8447	9049	9661	10273	10939	11617	12253	12889	13513	14177	14827
7213	7829	8461	9059	9677	10289	10949	11621	12263	12893	13523	14197	14831
7219	7841	8467	9067	9679	10301	10957	11633	12269	12899	13537	14207	14843
7229	7853	8501	9091	9689	10303	10973	11657	12277	12907	13553	14221	14851
7237	7867	8513	9103	9697	10313	10979	11677	12281	12911	13567	14243	14867
7243	7873	8521	9109	9719	10321	10987	11681	12289	12917	13577	14249	14869
7247	7877	8527	9127	9721	10331	10993	11689	12301	12919	13591	14251	14879
7253	7879	8537	9133	9733	10333	11003	11699	12323	12923	13597	14281	14887
7283	7883	8539	9137	9739	10337	11027	11701	12329	12941	13613	14293	14891
7297	7901	8543	9151	9743	10343	11047	11717	12343	12953	13619	14303	14897
7307	7907	8563	9157	9749	10357	11057	11719	12347	12959	13627	14321	14923
7309	7919	8573	9161	9767	10369	11059	11731	12373	12967	13633	14323	14929
7321	7927	8581	9173	9769	10391	11069	11743	12377	12973	13649	14327	14939
7331	7933	8597	9181	9781	10399	11071	11777	12379	12979	13669	14341	15121
7333	7937	8599	9187	9787	10427	11083	11779	12391	12983	13679	14347	15131
7349	7949	8609	9199	9791	10429	11087	11783	12401	13001	13681	14369	15137
7351	7951	8623	9203	9803	10433	11093	11789	12409	13003	13687	14387	15139
7369	7963	8627	9209	9811	10453	11113	11801	12413	13007	13691	14389	15149
7393	7993	8629	9221	9817	10457	11117	11807	12421	13009	13693	14401	15161
7411	8009	8641	9227	9829	10459	11119	11813	12433	13033	13697	14407	15173
7417	8011	8647	9239	9833	10463	11131	11821	12437	13037	13709	14411	15187
7433	8017	8663	9241	9839	10477	11149	11827	12451	13043	13711	14419	15193
7451	8039	8669	9257	9851	10487	11159	11831	12457	13049	13721	14423	15199
7457	8053	8677	9277	9857	10499	11161	11833	12473	13063	13723	14431	15217
7459	8059	8681	9281	9859	10501	11171	11839	12479	13093	13729	14437	15227
7477	8069	8689	9283	9871	10513	11173	11863	12487	13099	13751	14447	15233
7481	8081	8693	9293	9883	10529	11177	11867	12491	13103	13757	14449	15241
7487	8087	8699	9311	9887	10531	11197	11887	12497	13109	13759	14461	15259
7489	8089	8707	9319	9901	10559	11213	11897	12503	13121	13763	14479	15263
7499	8093	8713	9323	9907	10567	11239	11903	12511	13127	13781	14489	15269
7507	8101	8719	9337	9923	10589	11243	11909	12517	13147	13789	14503	15271
7517	8111	8731	9341	9929	10597	11251	11923	12527	13151	13799	14519	15277

ALGEBRA

In engineering, manufacturing, and industrial applications, physical laws govern the behavior of all quantities. Algebraic formulas (equations) are the models for these laws. They usually consist of algebraic expressions, the most common being polynomials, rational expressions, and radicals. Most of the formulas used in this *Handbook* contain one or more of these. This section gives a foundation for understanding the algebra indispensable to solving equations.

Definitions.—The vocabulary of algebra extends to all mathematics. The essential definitions are given here.

Operation: Addition, subtraction, multiplication, division, root-taking, raising to a power, taking a logarithm.

Inverse Operation: An operation that reverses another operation. Addition and subtraction are inverse operations, as are multiplication and division. Taking the nth root is the inverse of raising a number to a power. Finding an antilogarithm is the inverse of finding a logarithm.

Constant: A known quantity, either a number standing alone or a letter that is assumed to be given or known in an application. In $5x + 14$, 14 is the constant. Usually, the letters a, b, and c are used to represent constants, as in the linear equation $ax + by = c$. Note: e and π are commonly seen constants.

Variable: An unknown quantity, represented by a letter such as n, x, y, x, t. Note: e and π are not variables.

Exponent: The power to which a variable or number is raised.

Monomial: A single variable or number or a product of numbers and variables. Examples: $5, x, -4y^2, 12xy^2z^3$. Exponents in monomials are whole numbers, 0, 1, 2, 3, . . . , so $x^{-1} = 1/x$ and $x^{1/2} = \sqrt{x}$ are not monomials.

Coefficient: The numerical factor in a term. Examples (in bold): **5**x, **16**n, –**2**r. The coefficient of a variable standing alone is understood to be 1, for example, $x = 1x$; the coefficient of $-x$ is -1.

Term: Monomials are terms, but so are expressions that are not monomials: $1/x$, $\sqrt{x}$, $8x^{1/3}$, $\log x$, and so on.

Like Terms: This usually refers to monomials with the same variable and exponent, and having any real number coefficients, such as x and $7x$; $2n^2$ and $n^2/4$; $2rst/5$ and $14rst$, and so on. Any constant a can be written as ax^0, so all constants are like terms. But $x^{1/2}$ and $4x^{1/2}$ also are like terms.

Expression: Numbers and variables with *operators* (addition: +, multiplication: × or ·, $\sqrt{\ }$ etc.).

Equation: Two expressions set equal to one another with the equal sign = .

Examples: $5x = x^2 - 6$; $\sqrt{3x} = 14$. Solving equations for the unknown is the foundation of algebra.

Inequality: Two expressions set against one another by $>, <, \geq, \leq$, or $\neq$.

Evaluating Algebraic Expressions.—An expression is *evaluated* by substituting given values of the variable. For example, $x^2 - 2x + 7$ evaluated at $x = -3$ is $(-3)^2 - 2(-3) + 7 = 9 + 6 + 7 = 22$.

Another example, $\sqrt{1 - x^2}$ evaluated at $x = \frac{1}{3}$ is $\sqrt{1 - (\frac{1}{3})^2} = \sqrt{1 - \frac{1}{9}} = \sqrt{\frac{8}{9}}$.

Combining Like Terms.—Like terms are added and subtracted by combining their coefficients and leaving the rest of the term as is. For example:

$$3x + x - 7x = -3x \qquad 2n^2 + \frac{n^2}{4} - n^2 = \frac{5n^2}{4} \qquad 2rst - 14rst + rst = -11rst$$

Solving an Equation for an Unknown.—Solving an equation for an unknown (say, x) requires isolating x from the other terms. This is accomplished by applying techniques of *inverse operations*, *combining like terms*, and *factoring*, as explained throughout *ALGEBRA*, *GEOMETRY*, and *TRIGONOMETRY: SOLUTION OF TRIANGLES*.

The simplest equations contain only one or two variables, such as linear equations and other polynomials in two dimensions. Others may contain many variables, such as formulas in physics and engineering. A *formula* is an equation that determines a physical quantity based on other known quantities. An example is area of a rectangle, $A = lw$, where l is length and w is width in like units. Another is horsepower transmitted by belting, $P = SVW/33{,}000$, where S, V, and W are, respectively, working stress of the belt, belt velocity, and belt width, all in appropriate units.

The following examples give an overview of equation solving with general terms A, B, C, and D. The same processes apply for solving polynomial and other algebraic equations for an unknown.

Solving by Adding or Subtracting:

Given: $B - C = A - D$

To solve for A: $B - C + D = A$, that is, $A = B - C + D$

To solve for B: $B = A - D + C$

To solve for C: $B = A - D + C \rightarrow B - A + D = C$, that is, $C = B - A + D$

To solve for D: $B - C + D = A \rightarrow D = A - B + C$

In the last two, C and D are moved first so they do not have a minus sign before them in the answer. But *keeping in mind that subtracting a term is the same as adding its negative* (see *Integers (Signed Numbers)* on page 3), another way to solve for C or D is given below. The subtraction "minus C" is treated as "negative C." Then, multiplying through by –1, all signs change but equality is maintained, leaving C:

Solve for C: $B - C = A - D \rightarrow -C = A - D - B \rightarrow (-1)(-C) = (-1)(A - D - B) \rightarrow C = -A + D + B$

Solve for D: $B - C = A - D \rightarrow B - C - A = -D \rightarrow (-1)(B - C - A) = (-1)(-D) \rightarrow -B + C + A = D$, or $D = A - B + C$

Changing the sign of each term by multiplying through by –1 is a common technique.

Solving by Multiplying or Dividing: A variable that is a factor or a divisor within a term is isolated using division (*multiplying by the reciprocal*) or multiplication (*cross-multiplying*).

Example: In the equation $\frac{AB}{C} = D$, each variable is isolated using multiplication.

To isolate A, multiply each side by the reciprocal of $\frac{B}{C}$:

$$\frac{C}{B} \times \frac{AB}{C} = \frac{C}{B} \times D \rightarrow A = \frac{C}{B} \times \frac{D}{1} = \frac{CD}{B}, \text{ so } A = \frac{CD}{B}$$

To isolate B, multiply each side by the reciprocal of $\frac{A}{C}$:

$$\frac{C}{A} \times \frac{AB}{C} = \frac{C}{A} \times D \rightarrow B = \frac{C}{A} \times \frac{D}{1} = \frac{CD}{A}, \text{ so } B = \frac{CD}{A}$$

To isolate C, multiply each side by C: $C \times \frac{AB}{C} = C \times D \rightarrow AB = CD$

Then divide by D (multiply by the reciprocal of D):

$$\frac{AB}{1} \times \frac{1}{D} = \frac{CD}{1} \times \frac{1}{D} \rightarrow \frac{AB}{D} = C, \quad \text{so } C = \frac{AB}{D}$$

Equations with more complexity require more steps to solve for an unknown.

Examples:

Solve $\frac{A+B}{C} = D$ for A:

Multiply both sides by C	$\frac{A+B}{C} \cdot C = D \cdot C \rightarrow A + B = DC$
Subtract B from both sides	$A + B - B = DC - B \rightarrow A = DC - B$ or $CD - B$

Solve $\frac{A+B}{C} = B - D$ for D:

Add D to both sides	$\frac{A+B}{C} + D = B$
Subtract $\frac{A+B}{C}$ from both sides	$D = B - \frac{A+B}{C}$

Solve $\frac{A+B}{C} = B - A$ for A:

Add A to both sides	$\frac{A+B}{C} + A = B$	
Multiply both sides by C	$C\left(\frac{A+B}{C} + A\right) = CB$	
Distribute C	$A + B + CA = CB$	Like terms A and CA are isolated from B
Subtract B from both sides	$A + CA = CB - B$	Since A and CA cannot be combined, A must be factored out
Factor out A	$A(1 + C) = CB - B$	
Divide both sides by $1 + C$	$A = \frac{CB-B}{1+C}$	

Solving an equation for an unknown is the basic technique for working with formulas.

Rearrangement and Transposition of Terms in Formulas

A *formula* is a rule for a calculation expressed by using letters and signs, instead of writing out the rule in words. By this means, it is possible to condense, in a small space, the essentials of long and cumbersome rules.

As an example, the formula for horsepower transmitted by belting may be written

$$P = SVW/33000$$

where P = horsepower transmitted; S = working stress of belt per inch of width in pounds; V = velocity of belt in feet per minute; W = width of belt in inches; and, 33,000 = a constant that is part of the formula for horsepower with units of hp/(lb-in^2/min).

If the working stress S, velocity V, and width W are known, horsepower can be found directly from this formula by inserting the given values. For example, if $S = 33$, $V = 600$, and $W = 5$. Then,

$$P = 33 \times 600 \times 5/33000 = 3 \text{ hp}$$

Assume that horsepower P, stress S, and velocity V are known, and that the width of belt W is to be found. The formula must then be rearranged so that the symbol W will be alone on one side of the equation. This is accomplished by isolating W by moving the other variables and the number to the other side of the equation:

From $P = \frac{SVW}{33,000}$, multiply both sides by 33,000: $\quad 33,000P = SVW$

Divide both sides by SV: $\quad \frac{33,000P}{SV} = W$, or $W = \frac{33,000P}{SV}$

Algebraic Operations

Algebraic operations rely on the symbols and properties of real numbers, reviewed below from *ARITHMETIC* and used throughout this section. Key skills include exponent use (for both powers and roots), polynomial operations, and solving equations for a variable (unknown). The same skills carry over to areas of geometry, trigonometry, and calculus, where a strong foundation in algebra is essential.

Properties of Monomials and Exponents

$$a + \cdots + a = na \quad (n \text{ terms}) \qquad a + 0 = a \qquad a + (-a) = 0 \qquad a \times b = ab$$

$$a \times a \times \cdots \times a = a^n \quad (n \text{ factors}) \qquad a^n a^m = a^{(n+m)} \qquad \frac{a^n}{a^m} = a^{(n-m)} \qquad (a^m)^n = a^{mn}$$

$$a^{m/n} = (a^{1/n})^m \qquad a^{-n} = \frac{1}{a^n} \qquad \frac{1}{a^{-n}} = a^n \qquad a^0 = 1 \ \ (\text{for } a \neq 0)$$

Examples:

$$a + a = 2a \qquad -3a - a = -4a \qquad 4a \times 3a = 12aa = 12a^2$$

$$a^2 a^3 = a^{2+3} = a^5 \qquad (a^2)^3 = a^{2 \times 3} = a^6$$

$$a^{-3} = \frac{1}{a^3} = \left(\frac{1}{a}\right)^3 \qquad (ab)^2 = a^2 b^2 \qquad \left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3} \qquad \frac{a^4}{a^3} = a^{4-3} = a$$

Properties of Radicals

$$\sqrt{a} \times \sqrt{a} = a \qquad \sqrt{a} \times \sqrt{b} = \sqrt{ab} \qquad \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

$$(\sqrt[n]{a})^n = a \qquad \sqrt[n]{a^m} = (\sqrt[n]{a})^m = a^{m/n}, \text{ unless } a < 0, m \text{ and } n \text{ are both even}$$

$$\sqrt[n]{\frac{1}{a}} = \frac{1}{\sqrt[n]{a}} = a^{-1/n} \qquad (\sqrt{a} + \sqrt{b})^2 = (\sqrt{a} + \sqrt{b})(\sqrt{a} + \sqrt{b}) = a + 2\sqrt{ab} + b$$

Examples:

$$\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = (\sqrt[3]{a})^3 = a \qquad \sqrt[3]{ab} = \sqrt[3]{a} \times \sqrt[3]{b} \qquad \sqrt[3]{\frac{a}{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}$$

$$\sqrt[3]{a^2} = (\sqrt[3]{a})^2 = a^{2/3} \qquad \sqrt[3]{\frac{1}{a}} = \frac{1}{\sqrt[3]{a}} = \frac{1}{a^{1/3}} = a^{-1/3} \qquad \sqrt[4]{\sqrt[3]{a}} = \sqrt[4 \times 3]{a} = \sqrt[3]{\sqrt[4]{a}} \qquad (a^{1/4})^{1/3}$$

$$\sqrt{x} + 9\sqrt{x} - 4\sqrt{x} = (1 + 9 - 4)\sqrt{x} = 6\sqrt{x} \qquad \sqrt{x^2} = x \qquad \sqrt{x^3} = \sqrt{x^2 x} = \sqrt{x^2}\sqrt{x} = x\sqrt{x}$$

$$\sqrt{100x} = \sqrt{100}\sqrt{x} = 10\sqrt{x} \qquad \sqrt[5]{x^3} = \left(\sqrt[5]{x}\right)^3 = x^{5/3} \qquad \sqrt{20x^3} = \sqrt{4 . 5 . x^2 . x} = \sqrt{4x^2}\sqrt{5x} = 2x\sqrt{5x}$$

$$4\sqrt{5x}\left(\sqrt{x} + 3x\right) = \sqrt{5x \cdot x} + 12x\sqrt{5x} = 4\sqrt{5x^2} + 12x\sqrt{5x} = 4x\sqrt{5} + 12x\sqrt{5x}$$

$$\sqrt{\frac{36}{x^4}} = \frac{\sqrt{36}}{\sqrt{x^4}} = \frac{6}{x^2} \qquad \sqrt{x+1} \text{ cannot be simplified}$$

Polynomials

Polynomials in a single variable are expressions of the form $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x^1 + a_0$, where *n is a non-negative integer*, and the coefficients $a_n, a_{n-1}, \ldots, a_0$ are real numbers (the subscripts are simply labels that correspond to the variables (x^n, x^{n-1}, . . .). n is the *degree* (or *order*) of the polynomial; a_n is the *leading coefficient*; a_0 is the *constant coefficient*. A *first-degree polynomial* has $n = 1$; a *second-degree* polynomial has $n = 2$, and so on.

Example: $\frac{x}{3} + \frac{1}{6}$ is degree 1, $a_1 = \frac{1}{3}$ is the leading coefficient, $a_0 = \frac{1}{6}$ is the constant coefficient.

Example: $4x^2 - 7$ is degree 2, $a_2 = 4$ is the leading coefficient, $a_0 = -7$ is the constant coefficient.

Example: $x^3 + 5x^2 - x + \frac{1}{3}$ is degree 3, and the coefficients are $a_3 = 1, a_2 = 5, a_1 = -1$, and $a_0 = \frac{1}{3}$.

A *monomial* is a single-term polynomial. For example, $8x^5$ is degree 5 with coefficient 8. A constant is a polynomial, by the definition. Its degree is 0. For example, 19 is a 0-degree polynomial, since it can be written as $19x^0$. A *binomial* is a polynomial with two terms, such as $x + 9$ and $5 - x^2$. A *trinomial* has three terms, such as, $x^2 + x - 6$.

Finally, second-degree polynomials are called *quadratic* polynomials. They are important because they model so many processes in engineering and other technical and scientific fields.

Operations on Polynomials.—Polynomials can be added, subtracted, or multiplied ("expanded"), with the result being another polynomial. If polynomials are divided, the result is a *rational expression*. Polynomials also may be *factored*. That is, they may be written as a product of lower-degree polynomials.

Combining (Adding and Subtracting) Polynomials: Two or more polynomials are added by combining like terms. For example:

$$\left(x^4 + 6x^3 - 3x^2 - 5x - 11\right) + \left(x^3 - 12x^2 + x + 28\right) = x^4 + 7x^3 - 15x^2 - 4x + 17$$

Multiplying (Expanding) Polynomials: Taking the product of two or more polynomials relies on the distributive property of multiplication over addition (or subtraction): $a(b + c) = ab + ac$, where the rules of exponents are followed (see *Properties of Monomials and Exponents* on page 22).

Examples of simple distributive case:

$$2x\left(8x^4 + 3x^2 - 6\right) = 2 \cdot 8x^{1+4} + 2 \cdot 3x^{1+2} - 2 \cdot 6x = 16x^5 + 6x^3 - 12x$$

Note: This technique also is used for multiplying other algebraic expressions, such as radical (root) and rational expressions. Such terms are not polynomials, since their exponents are other than non-negative integers.

Examples of Distributive Property for Root and Rational Expressions:

$$2x^{1/2}\left(7x + 3\right) = 14x^{1/2+1} + 6x^{1/2} = 14x^{3/2} + 6x^{1/2}$$

$$\frac{1}{x}\left(x^3 - 7x^2 + 3x\right) = x^{-1+3} - 7x^{-1+2} + 3x^{-1+1} = x^2 - 7x^1 + 3x^0 = x^2 - 7x + 3$$

"FOIL" is a version of the distributive property in which two binomials are multiplied. FOIL stands for **F**irst **O**uter **I**nner **L**ast, the order in which terms are multiplied. The first is ac; outer is ad, inner is bc, and last is bd.

FOIL multiplication: $(a + b)(c + d) = ac + ad + bc + bd$

Examples:

$$(x-6)(x+1) = x \cdot x + x \cdot 1 - 6 \cdot x - 6 \cdot 1 = x^2 + 1x - 6x - 6 = x^2 - 5x - 6$$

$$(3x^2 + 2x)(4x^3 - 5) = 3x^2 \cdot 4x^3 - 5 \cdot 3x^2 + 2x \cdot 4x^3 + 2x \cdot (-5) = 12x^5 - 15x^2 + 8x^4 - 10x$$

$$(x-1)(x+1) = x \cdot x + x \cdot 1 - 1 \cdot x - 1 \cdot 1 = x^2 + 1x - 1x - 1 = x^2 - 1 \text{ (the middle terms drop out)}$$

$$(x+1)^2 = (x+1)(x+1) = x^2 + x + x + 1 = x^2 + 2x + 1$$

In the first and third examples, the products of the outer and inner terms are like terms, so they are combined. The third example is a *special product* (in this case, the difference of squares). The fourth is an example of raising a polynomial to an exponent.

Raising a Polynomial to an Exponent: Consider first $(a+b)^2$, which is $(a+b)(a+b)$, and this expands by FOIL to $a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$. In general, an expansion like $(a+b)^2$, $(a-b)^2$, $(a+b)^3$, $(a-b)^3$ is called a *binomial expansion*, $(a+b)^n$. For example:

$$(a+b)^n = \underbrace{(a+b)(a+b)(a+b)\cdots(a+b)}_{n \text{ factors}}$$

Caution: A common but *serious* mistake is distributing the exponent to each term in the parentheses:

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2 \quad \textit{NOT} \quad a^2 + b^2$$

Factoring Polynomials.—The result of two or more polynomials being multiplied is a higher-degree polynomial. *Factoring a polynomial* breaks it into its lesser-degree factors. This section explains how to factor a polynomial, a skill needed for solving equations and graphing functions (see Graphs of Functions on page 30). The categories are: dividing by a *common factor* (i.e., distributive property in reverse); factoring by *reverse FOIL* (with leading coefficient of 1 and otherwise); factoring *special products*.

Common Factors: Each term is divided by the *greatest common factor* (GCF) and written as shown: $ab + ac = a(b+c)$

Examples of GCF factoring:

$$6x^3 + 2x^2 - 10x = 2x(3x^2 + x - 5) \qquad \text{GCF}$$

$$mn + m^2n - mn^2 = mn(1 + m - n) \qquad \text{GCF}$$

$$-7abc - 21bc^2 + 4bc = -bc(7a + 21c - 4) \qquad \text{GCF}$$

Note: The variable in the GCF is the one with the lowest exponent in common to all terms. If two terms have a GCF, but the others do not, then that factor cannot be pulled out. And, if the leading coefficient is negative (third example), it is customary to factor out –1 in the GCF redundant. This is so the expression in parentheses has a positive leading coefficient, which makes it easier to factor further.

Reverse FOIL of Form $x^2 + bx + c$: The basic technique is demonstrated for a second-degree trinomial with a leading coefficient of 1. The task is to factor $x^2 + bx + c$ as $(x + \square)(x + \square)$, using only integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ in the boxes.

Example: Consider $x^2 + 4x + 3$. To factor it as $(x + \square)(x + \square)$, the integers must have a product of +3 (the last term of $x^2 + 4x + 3$) and a sum of +4 (the middle term). 1 and 3 are correct, as verified by FOIL:

$$x^2 + 4x + 3 = (x+1)(x+3).$$

As long as the leading coefficient is 1, reverse FOIL works. If one or both operations in the trinomial are negative, the process is the same, but some trial and error may be needed, as shown in the examples below.

Example: Three similar trinomials, only two are factorable:

$x^2 + 2x - 3 = (x+3)(x-1)$, since $(3)(-1) = -3$ and $3 + -1 = 2$

$x^2 - 2x - 3 = (x-3)(x+1)$, since $(-3)(1) = -3$ and $-3 + 1 = -2$

$x^2 - 2x + 3$ Does not factor over the real numbers, since no two integers have a product of $+3$ and a sum of -2

Examples: Factoring by reverse FOIL, along with the explanation:

$x^2 - 8x - 20 = (x-10)(x+2)$ The two integers whose product is –20 and whose sum is –8 are –10 and 2.

$x^2 - 21x + 20 = (x-20)(x-1)$ The two integers whose product is 20 and whose sum is –21 are –20 and –1.

Reverse *FOIL for polynomials of Form* $ax^2 + bx + c$: If the leading coefficient is other than 1, reverse FOIL works, but a different tactic is needed. Sometimes called "magic factoring," the procedure is shown in this example:

$$3x^2 + 13x + 12, \quad a = 3, b = 13, c = 12$$

Step 1: $ac = (3)(12) = 36$	
Step 2: Write all factor pairs of $ac = 36$:	$1,36; 2,18; 3,12; 4,9; 6,6$
Step 3: Choose the factor pair whose sum is $b = 13$:	$4,9$
Step 4: Rewrite polynomial with middle term $13x$ as $4x + 9x$:	$3x^2 + 4x + 9x + 12$
Step 5: Group first two terms and second two terms in parentheses:	$(3x^2 + 4x) + (9x + 12)$
Step 6: Take the common factor out of each group:	$x(3x + 4) + 3(3x + 4)$
Step 7: Take the common *binomial* factor out of each large term:	$(3x + 4)(x + 3)$ Correct factorization
Step 8: Check the factorization found:	$3x^2 + 9x + 4x + 12 = 3x^2 + 13x + 12$

Special Products: Certain binomials are factored according to formulas, which can be checked by multiplying. The *difference of squares* is perhaps the most important because it comes up so often in applications, as do many quadratic (second-degree) polynomials. It comes from multiplying the *conjugate pair* of binomials, $(a + b)$ with $(a - b)$ to get $a^2 - b^2$.

Difference of squares:	$a^2 - b^2 = (a + b)(a - b)$
Difference of cubes:	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
Sum of cubes:	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Square of a sum:	$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$
Difference of a sum:	$(a - b)^2 = (a - b)(a - b) = a^2 - 2ab + b^2$

Note: *The sum of squares* $a^2 + b^2$ *is not a factorable binomial over the set of real numbers*. However, it is factorable over the set of complex numbers (see *Complex Numbers* on page 54).

Examples: Several special product factorizations are shown:

Difference of squares: $a^2 - b^2$	Differerence of cubes: $a^3 - b^3$	Sum of cubes: $a^3 + b^3$
$x^2 - 81 = (x+9)(x-9)$	$x^3 - 1 = (x-1)(x^2 + x + 1)$	$x^3 + 1 = (x+1)(x^2 - x + 1)$
$100 - 49x^2 = (10 + 7x)(10 - 7x)$	$8 - x^3 = (2 - x)(4 + 2x + x^2)$	$27 + x^3 = (3 + x)(9 - 3x + x^2)$

Factorization of all other cases is done using the *quadratic formula*, as shown in the next section.

Equation Solving

An equation is a statement of equality between two expressions, such as one monomial set equal to another, like $5x = 105$. The unknown, or variable, is frequently designated by x. Other unknowns (if any) are designated by letters also usually selected from the end of the alphabet: y, z, u, t, etc.

Equations, like expressions, have a degree. A *first-degree* equation is one in which the variable is raised to the first power, as in $3x = 9$. A *second-degree* equation, also called a *quadratic* equation, is one in which the highest power of the variable is two; for example, $x^2 + 3x = 10$.

Solving a first-degree equation requires isolating the unknown. In the example below, x is the unknown variable. To get x alone (to isolate x), constants are combined on one side of the equation and variable terms are combined on the other. The steps are:

Given:	$10x - 14 = 8 - 2x$
Add $2x$ to both sides:	$10x - 14 + 2x = 8 - 2x + 2x$
	$12x - 14 = 8$
Add 14 to both sides:	$12x - 14 + 14 = 8 + 14$
	$12x = 22$
Divide by 12 (multiply by 1/12):	$12x/12 = 22/12$
Simplify:	$x = 22/12 = 11/6$

Any answer can be checked by substituting it into the original equation to see that it satisfies it.

Solving a System of Linear Equations.—More involved than solving a single-variable equation is the process of solving a *system* of linear equations. A simple linear system represents two lines in the plane that behave in one of three ways: they *intersect at one point,* in which case they have a *unique solution*, (x, y); they intersect everywhere—that is, they are *collinear*—so all points (x, y) of one line satisfy the other; or they are *parallel* and thus intersect nowhere, hence, there is no solution.

The methods for solving a system of linear equations are *substitution* and *elimination*, as shown next.

Substitution: In this method, one of the variables is expressed in terms of the other variable by isolating it. This expression is then substituted into the second equation, converting it to a single-variable equation. It is solved for this variable, and the solution is substituted back into the either of the original two equations to find the value of the other variable.

Example (Unique Solution): Find the ordered pair (x, y) that satisfies the system of equations:

$$2x + y = 7$$
$$x - 2y = -4$$

First, solve either equation for one variable in terms of the other. Say, solve the second equation for x:

$$x = 2y - 4$$

Then, substitute this expression for x into the first equation and solve it for y:

$$2(2y - 4) + y = 7 \quad \rightarrow \quad 4y - 8 + y = 7 \quad \rightarrow \quad 5y = 15 \quad \rightarrow \quad y = 3$$

Finally, substitute $y = 3$ into the second equation: $x = 2(3) - 4 = 6 - 4 = 2$. The solution (that is, the point at which the lines intersect) is (2, 3).

Example (Infinite Solutions): Find the ordered pair (x, y) that satisfies the system of equations:

$$7x - y = 3$$
$$14x - 2y = 6$$

The first equation is rearranged to $y = 7x - 3$. Substituting into the second equation yields: $14x - 2(7x - 3) = 6 \rightarrow 14x - 14x + 6 = 6 \rightarrow 6 = 6$.

If a system results in $a = a$, then the lines are the same, or collinear. All of the points on either line are solutions to (points on) the other line. The indication that lines are collinear is that one is a multiple of the other, term by term.

Example (No Solution): Parallel lines have the same slope, and so they have no point of intersection. This is seen in the substitution of one equation's variable into the other equation, giving a false statement, such as 1 = 3. An example is the system $y = -x + 4$ and $y = -x - 1$. Substitution of the first into the second gives $-x + 4 = -x - 1$, or $4 = -1$. This clearly is not true, hence the system has no solution.

A final way to find the solution to a linear system is to put the coefficients of the equations into the following formulas and to solve for x and y, as follows:

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Then,

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \qquad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

Example:

$$3x + 4y = 17$$
$$5x - 2y = 11$$

$$x = \frac{(17)(-2) - (11)(4)}{(3)(-2) - (5)(4)} = \frac{-34 - 44}{-6 - 20} = \frac{-78}{-26} = 3$$

The value of y can now be most easily found by inserting the value of x in one of the equations:

$$(5)(3) - 2y = 11, \qquad 2y = 15 - 11 = 4, \qquad y = 2$$

Checking the solution by putting these values into the original system shows that (3,2) is the solution of this linear system.

Solving a Second-Degree (Quadratic) Equation.—A second-degree equation is also called a *quadratic equation*. To *solve* a second-degree equation is to find the x values (or points) at which the parabola intersects the x-axis. These are the *roots* or *zeros* of the parabola.

There are several ways to solve a quadratic equation for its roots. If an equation can be factored, then each factor is set equal to zero and the solution is thus found. This is according to the *zero property of multiplication*, which states that if $AB = 0$, then either $A = 0$ or $B = 0$.

If the equation is not readily or obviously factored (or even if it is), the *quadratic formula* can be used to find the roots, as explained in the next section, *Using the Quadratic Formula*.

The simplest quadratic equation has the form $x^2 = c$, where c is a constant. Solving this entails simply taking the square root of both sides:

$x^2 = c$, therefore, $x = \sqrt{c}$ and $-\sqrt{c}$. For example, $x^2 = 36$ has two solutions, $x = 6$ and $x = -6$. If the quadratic has the form, like $ax^2 = c$, the solution is also straightforward, $x^2 = a/c$, and so $x = \pm\sqrt{a/c}$.

Note: The square root of a number as it stands alone is understood to be its positive root; that is, the square root of 9 is 3. But *in an equation* the solution includes both positive and negative roots. This makes sense because the equation represents a parabola, which can intersect the x-axis in two places, its two roots. The other two possibilities occur when the parabola (a quadratic function) intersects the x-axis once (touches it) or does not intersect it. All three possibilities are revealed in solutions to the quadratic equation.

Designating the roots of a quadratic equation (or any polynomial) by r, $x^2 = c$ can be regarded as $x^2 = r^2$, so $x = r$ and $-r$, the roots of the parabola. Another way to rewrite $x^2 = r^2$

is $x^2 - r^2 = 0$, and then factor as the difference of squares (see *Polynomials* on page 23): $(x+r)(x-r)=0$, hence, $x = -r$ and r.

Verifying this to be the factorization of the difference of squares is simply a matter of applying the distributive property of multiplication over addition to each term:

$$(x+r)(x-r) = xx - xr + xr - rr = x^2 - r^2$$

As previously explained, this process is called expanding by FOIL, for the **F**irst, **O**uter, **I**nner, and **L**ast terms of each binomial, which are multiplied in that order to get the product. From the zero property of multiplication,

$$(x+r)(x-r)=0 \quad \text{implies} \quad x+r=0 \quad \text{or} \quad x-r=0, \text{thus } x=-r \quad \text{or} \quad x=r.$$

If a quadratic cannot be factored as a difference of squares, then factorization is approached as shown in *Factoring Polynomials* on page 24. Given a quadratic equation in the form $ax^2 + bx + c = 0$, first obtain the product ac from the coefficients a and c; then determine two numbers whose product is ac and whose sum is b.

Example: Find the solution to $x^2 - 5x + 6 = 0$ by factoring.

In this example, $a = 1, b = -5, c = 6$ and $ac = (1)(6) = 6$. The factors of 6 whose sum is –5 are –2 and –3. The equation is factored as $x^2 - 5x + 6 = (x-2)(x-3) = 0$. Then by the zero property of multiplication, the roots of the equation are $x = 2$ and $x = 3$. The parabola intersects the x-axis at these two values of x.

A more difficult example has a leading coefficient of x other than 1.

Example: Factor $8x^2 + 22x + 5 = 0$ and find the values of x that satisfy the equation.

Solution: Here, $a = 8, b = 22$, and $c = 5$. Therefore, $ac = 8 \times 5 = 40$, and ac is positive, so we are looking for two factors of ac, namely f_1 and f_2, such that $f_1 + f_2 = 22$.

The possible combinations of numbers with product of 40 are 20 and 2, 8 and 5, 4 and 10, and 40 and 1. The requirements that satisfy a sum of 22 are 20 and 2, since $20 \times 2 = 40$ and $20 + 2 = 22$. Hence:

$$\begin{aligned} 8x^2 + 22x + 5 &= 0 \\ 8x^2 + 20x + 2x + 5 &= 0 \\ 4x(2x+5) + 1(2x+5) &= 0 \\ (2x+5)(4x+1) &= 0 \end{aligned}$$

On the second line, the common factor of $4x$ in the first two terms is factored out, so the common binomial factor of $2x + 5$ is then apparent, to be factored out of the larger terms. Checking the answer is a matter of simply remultiplying the factors to produce the original expression.

Because the product of the two factors equals zero, each of the factors also equals zero. Thus, $2x + 5 = 0$ and $4x + 1 = 0$. Rearranging and solving, $x = -5/2$ or $x = -1/4$.

Example: Factor $8x^2 + 3x - 5 = 0$ and find the solutions of the equation.

Solution: Here $a = 8, b = 3, c = -5$, and $ac = 8 \times (-5) = -40$. The required numbers must have a product of –40 and a sum of 3.

As in the previous example, the possible combinations are 20 and –2, –20 and 2, –8 and 5, 8 and –5, 40 and –1, and –40 and 1. Only 8 and –5 satisfy the requirements because $8 \times (-5) = -40$, and $8 + (-5) = 3$. Notice that $3x$ in the first line is thus written as $8x - 5x$ in the second line, making it possible to rearrange and simplify the expression.

$$\begin{aligned} 8x^2 + 3x - 5 &= 0 \\ 8x^2 + 8x - 5x - 5 &= 0 \\ 8x(x+1) - 5(x+1) &= 0 \\ (x+1)(8x-5) &= 0 \end{aligned}$$

Solving, for $x + 1 = 0, x = -1$; and, for $8x - 5 = 0, x = 5/8$.

Solving by Completing the Square.—An equation of the form $x^2 + bx + c = 0$ can be turned into the square of a sum. The steps are:

- Move constant to the right side of the equal sign: $x^2 + bx = -c$
- Add $(b/2)^2$ to both sides: $x^2 + bx + (b/2)^2 = -c + (b/2)^2$
- Note the left side is the square of a sum, that is: $(x + b/2)^2$
- The right is a new constant, call it d: $-c + (b/2)^2 = d$
- The equation is thus converted to: $(x + b/2)^2 = d$
- Take the positive and negative square root of both sides: $x + \frac{b}{2} = \pm\sqrt{d}$
- Solve for x: $x = -\frac{b}{2} \pm \sqrt{d}$

Example: Complete the square to solve $x^2 + 4x - 3 = 0$.

- $x^2 + 4x = 3$
- $x^2 + 4x + (4/2)^2 = 3 + (4/2)^2$, that is, $x^2 + 4x + 4 = 3 + 4$
- Left side: $x^2 + 4x + 4 = (x + 2)^2$
- Right side: $3 + 4 = 7$
- So, $(x + 2)^2 = 7$
- $x + 2 = \pm\sqrt{7}$
- $x = -2 \pm\sqrt{7}$

Completing the square is more involved if the leading coefficient a is not 1. Then it is preferable to use the *quadratic formula*, which can be used for any value of a in a quadratic equation.

Using the Quadratic Formula.—The method of completing the square leads to the quadratic formula for finding the roots of equations with the form $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example: In the equation, $x^2 + 6x + 5 = 0$, $a = 1$, $b = 6$, and $c = 5$.

$$x = \frac{-6 \pm \sqrt{6^2 - (4)(1)(5)}}{(2)(1)} = \frac{(-6) + 4}{2} = -1 \quad \text{or} \quad \frac{(-6) - 4}{2} = -5$$

Example: A right triangle has a hypotenuse of 5 cm and one leg that is 1 cm longer than the other; find the lengths of the two legs.

Let x be the length of one leg and $x + 1$ be the length of the other; then by Pythagorean theorem (see page 57), $x^2 + (x + 1)^2 = 5^2$. Expanding this and setting all terms equal to zero gives $x^2 + x - 12 = 0$. Now referring to the basic formula, $ax^2 + bx + c = 0$, here, $a = 1$, $b = 1$, and $c = -12$. Substituting these values into the quadratic formula:

$$x = \frac{-1 \pm \sqrt{1 - (4)(1)(-12)}}{(2)(1)} = \frac{(-1) + 7}{2} = 3 \quad \text{or} \quad x = \frac{(-1) - 7}{2} = -4$$

Since only the positive value, 3, makes sense in this case, the lengths of the two sides are $x = 3$ cm and $x + 1 = 4$ cm.

Solving a Cubic Equation.—Just as quadratic equations may be simple to solve if they are of the form $x^2 = c$, a cubic equation of the form $x^3 = c$ also is simple to solve. But, the given equation has the form: $x^3 + ax + b = 0$ then one of the real roots is:

$$x = \left(-\frac{b}{2} + \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}\right)^{1/3} + \left(-\frac{b}{2} - \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}\right)^{1/3}$$

The equation $x^3 + px^2 + qx + r = 0$ may be rewritten in the form $x_1^3 + ax_1 + b = 0$ by substituting $x_1 - \frac{p}{3}$ for x in the given equation.

Functions

Functions are understood through both equations and graphs. Graphs are drawn on the (x, y)-coordinate system (or rectangular coordinate system), which is described fully in *Analytic Geometry* section of *GEOMETRY* starting on page 37.

A *function* consists of two sets (generally called X and Y) of numbers and a *rule* that assigns (or sends) each element (usually, a real number) x in X to a *unique* element (another real number) y in Y. "y is a function of x" is commonly expressed as $y = f(x)$. Countless physical processes are represented by functions. For example, displacement, s, is a function of time, t, so it is represented as $s(t)$. Velocity also is a function of time, $v(t)$.

The set X is called the *domain* of a function, which is the set of all real number values for which a function is *defined*. When the function acts on these numbers, the result is another real number y.

As an example, $y = f(x) = 2x + 5$ is a linear function whose domain is the set of all real numbers, since when 5 is added to the product of 2 and any real number x, the result is always a real number y. Lines are first-degree polynomials; in fact, the domain of *any* polynomial is the set of reals (see *Polynomials* on page 23).

Two more examples: $f(x) = 1/x$ is not defined at $x = 0$, since 1/0 is not a number, so its domain is the set of all real numbers except 0. Since negative numbers do not have square roots in the reals, the domain of $f(x) = \sqrt{x}$ is the set of non-negative real numbers.

In $y = f(x)$, x is called the *input* to the function. The y value resulting from x is the *output*, or *function value*. Since y depends on x, it is the *dependent variable*; x is the *independent variable*.

Interval Notation: Domain is often expressed in *interval notation*, indicating the portion of the number line that contains a function's valid input values. The interval notation for set of all reals is $(-\infty, \infty)$; for non-negative numbers, it is $[0, \infty)$; for positive numbers, it is $(0, \infty)$; and for all real numbers except 0, it is $(-\infty, 0) \cup (0, \infty)$. The domain of a polynomial is therefore $(-\infty, \infty)$; the domain of $f(x) = 1/x$ is $(-\infty, 0) \cup (0, \infty)$; and the domain of $f(x) = \sqrt{x}$ is $[0, \infty)$.

Graphs of Functions.—The *graph* of a function is drawn through the points (x, y) on the rectangular coordinate system (see page 37) so that the curve satisfies the equation $y = f(x)$. A graph depicts the relationship between x and y. The graph of a function will always pass the *vertical line test*, by which any vertical line intersects the graph at most once. Thus, any line other than a vertical line is expressible in function form, $f(x) = mx + b$, where the slope is m and the y-intercept is b. (The several *forms* of linear equations are explained in *Equation Forms of a Line* on page 39.) A parabola is represented by a polynomial function of the form $f(x) = ax^2 + bx + c$. All polynomials pass the vertical line test (Fig. 1a).

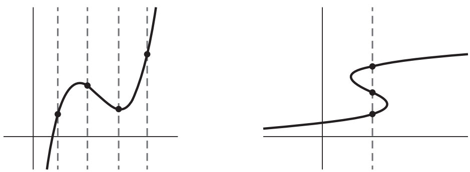


Fig. 1. (a) Passes Vertical Line Test—Represents a Function;
(b) Fails the Test—Does Not Represent a Function.

Sketches of Basic Functions: Sketches of basic functions and domain interval notation are shown in Fig. 2. It is obvious that each function passes the vertical line test.

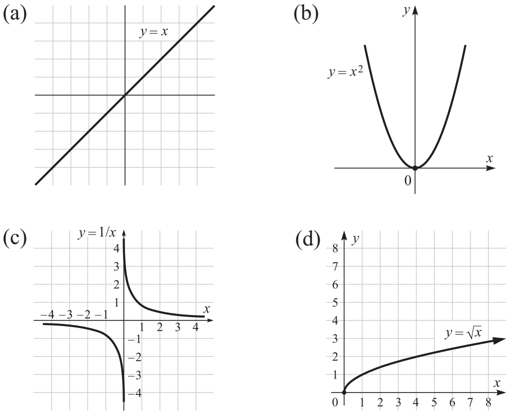


Fig. 2. (a) $f(x) = x, D_f = (-\infty, \infty)$; (b) $f(x) = x^2, D_f = (-\infty, \infty)$; (c) $f(x) = \sqrt{x}, D_f = [0, \infty)$; (d) $f(x) = 1/x, D_f = (-\infty, 0) \cup (0, \infty)$.

Logarithms

Logarithms are of value in many engineering and shop calculations because they make it possible to solve cumbersome and difficult problems that otherwise would require more complex mathematical methods. Because of this, "logs" (for short) have long been used to facilitate and shorten calculations involving multiplication, division, the extraction of roots, and obtaining powers of numbers. Since the advent of handheld calculators, however, logarithms are rarely used to do these basic operations. The *Guide* and *Logarithms* in the *ADDITIONAL MATERIAL* in the *Machinery's Handbook 32 Digital Edition* include explanations and examples of how the log tables are used for computation.

Log properties and principles are still necessary in many areas. Logarithmic growth and its inverse, exponential growth (and decay), are essential to investigating processes in technical fields and science, in general. The main principles and properties of logarithms are covered here, along with representative examples. In most cases, a calculator is used to arrive at the answers.

Meaning of Logarithm.—The logarithm of a given number is the exponent to which a stated base must be raised to produce the given number. A formulaic definition of logarithm is:

$$\log_b x = y \qquad \text{means} \qquad b^y = x$$

which is read, "The logarithm of x in base b is y; that is, b raised to the y power equals x." y is the logarithm and x is the *antilogarithm* ("antilog"). Base b is always greater than 1. The antilog must be positive, since a positive number b raised to any power cannot give zero or a negative number. Some examples:

$\log_2 8 = y$	means $2^y = 8$,	so $y = 3$
$\log_b 100 = 2$	means $b^2 = 100$,	so $b = 10$
$\log_3 x = -1$	means $3^{-1} = x$,	so $x = 1/3$

Properties of Logarithms.—The definition along with exponent rules covered previously lead to the *properties of logarithms*, given here with examples. When no base is shown, base 10 is implied. (See *Common Logarithms* on page 32.)

Property	*Example*
$\log_b 1 = 0$	$\log_3 1 = 0$
$\log_b b = 1$	$\log_8 8 = 1$
$\log_b(xy) = \log_b x + \log_b y$	$\log(2y) = \log 2 + \log y$
$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$	$\log\left(\frac{100}{3}\right) = \log 100 - \log 3$
$\log_b x^r = r\log_b x$	$\log_2 x^3 = 3\log_2 x$
$\log_b b^r = r$	$\log_5 5^{11} = 11$

Common Logarithms.—There are two standard systems of logarithms: *common* (base 10) and *natural* (base e, explained below). In general, $\log_{10}x$ is written simply as $\log x$. For example, $\log 100 = 2$ because $10^2 = 100$.

Examples:
$\log 15 = 1.176$, since $10^{1.176} \approx 15$
$\log 250 = 2.397$, since $10^{2.397} \approx 250$
$\log 4000 = 3.602$, since $10^{3.602} \approx 4000$

The log values seen here can be found by using either log tables or a scientific calculator. Because most logarithms are irrational numbers, the values given in any example and in the tables are approximations, rounded to several decimal places. When a calculator is used, the answer should be rounded to four or five decimal places.

The whole number part of a logarithm is called the *characteristic*; the decimal portion is called the *mantissa*. In the examples above, the characteristics are 1, 2, and 3, respectively, which correspond to the power of 10 of the antilog when it is written in scientific notation:

$$\log 15 = \log(1.5 \times 10^{\mathbf{1}}) = \log 1.5 + \log 10^{\mathbf{1}} = 0.176 + \mathbf{1} \approx 1.176$$
$$\log 250 = \log(2.5 \times 10^{\mathbf{2}}) = \log 2.5 + \log 10^{\mathbf{2}} = 0.397 + \mathbf{2} \approx 2.397$$
$$\log 4000 = \log(4.0 \times 10^{\mathbf{3}}) = \log 4.0 + \log 10^{\mathbf{3}} = 0.602 + \mathbf{3} \approx 3.602$$

The property $\log(ab) = \log a + \log b$ has been applied; the "$\log b$" portion (characteristic) is quickly determined, as it is simply the power of 10 (bold). The "$\log a$" portion (mantissa) is read from the log table, which gives logs of numbers from 1 to 10 up to a certain number of decimal places. If the log of a number less than 1 is to be found, again, the antilog is represented in scientific notation to get to the answer, only now the characteristic is negative, so a subtraction is involved. For example:

$$\log 0.63 = \log(6.3 \times 10^{\mathbf{-1}}) = (\log 6.3) + \log(\mathbf{-1}) = 0.799 + (\mathbf{-1}) \approx -0.201$$

Natural Logarithms.—In certain formulas and in some branches of mathematical analysis, use is made of the *natural* logarithm. The base of this system is given as e, which is the symbol for the irrational number that is approximately equal to 2.7182818284. (Recall that an irrational number cannot be represented by a repeating or terminating decimal.) e is the base of exponential growth phenomena such as populations and compound interest, among others. Though e was first conceptualized by John Napier (who developed logarithmic calculation) and developed further by Jacob Bernoulli, the use of e credits eighteenth-century Swiss mathematician Leonhard Euler (pronounced "oiler"), who developed mathematical analysis, in part, through his discovery of the so-called Euler identity, $e^{i\pi} = -1$.

It is conventional to write $\log_e x$ as "$\ln x$"; hence:

$$\ln x = y \text{ means } \log_e x = y, \text{ that is, } x = e^y$$

So, for example, $\ln e = 1$, since $e^1 = e$; $\ln 1 = 0$, since $e^0 = 1$; $\ln e^3 = 3\ln e = 3 \cdot 1 = 3$. And, the process of finding logarithm values in base e is the same as in base 10.

Example: To represent ln 0.239 as a sum of natural logs: $\ln(2.39 \times 10^{-1}) = \ln 2.39 + \ln 10^{-1} = \ln 2.39 + (-1)\ln 10$.

Logarithms in any base can be converted to another base by the formula: $\log_b x = \dfrac{\log_a x}{\log_a b}$

Logarithms of bases other than 10 and *e* were often converted to these bases, since the values were easy to look up on the common or natural log tables. The formula also can convert between the two most-often used bases:

Base *e* to base 10: $\ln x = \dfrac{\log x}{\log e} \approx \dfrac{\log x}{0.4343}$ so $\log x \approx 0.4343 \ln x$

Base 10 to base *e*: $\log x = \dfrac{\ln x}{\ln 10} \approx \dfrac{\ln x}{2.3026}$ so $\ln x \approx 2.3026 \log x$

Example: Convert ln 4 to log 4 using the conversion, $\log x \approx 0.4343 \ln x$:

$$\log 4 \approx 0.4343 \ln 4 \approx (0.4343)(1.3863) \approx 0.60207$$

Example: Express $\log_2 9$ in terms of the natural logarithm:

$$\log_2 9 = \frac{\log_e 9}{\log_e 2} = \frac{\ln 9}{\ln 2} \approx 3.170$$

Using Calculators to Solve Logarithms.—To find a common logarithm on a scientific calculator (handheld or online), the **log** key is used. To find a natural logarithm the **ln** key is used. Depending on the calculator used, either the number is entered before the log key is pressed, or log is entered before the number. The correct sequence can be seen if an error message results. For example, log 6 is found by this sequence on the typical scientific calculator:

Example: To find log 6, press in sequence, [6] [log] to get display

[0.77815125038 . . .]

To find the common antilog of a given number, the **10ˣ** key is used. The find the natural antilog, the **eˣ** key is used. This kind of problem is often asked as $\ln x = 4$, meaning $e^4 = x$, so one must know the definition of $\log_b x = y$ to get the answer by calculator.

Example: To find *x* in $\ln x = 4$, press in sequence, [4] [eˣ] get the display

[54.5980015003 . . .]

On calculators without the **10ˣ** and **eˣ** keys, the **xʸ** key enables, substituting 10 or *e* (2.718281 . . .) for *x* and the logarithm of the number sought for *y*. On some calculators, while the **log** and **ln** keys are used to find common and natural logarithms, the same keys in combination with the **INV**, or inverse, key are used to find the number corresponding to a given logarithm.

Solving an Equation Using Logarithms.—Solving exponential and logarithmic equations is possible because of the following properties of logs and exponents, which are true for *any* base:

If $x = y$, and $x, y > 0$, then $\log x = \log y$.   If $x = y$, then $a^x = a^y$ for $a > 0$, $a \neq 1$.

Both statements are true in the other direction, too:

If $\log x = \log y$, then $x = y$.   If $a^x = a^y$ then $x = y$.

Example 1: Find the square root of 754.

Solution: Let $x = \sqrt{754}$. Then $\log x = \log\sqrt{754} = \log 754^{1/2} = \frac{1}{2}\log 754 \approx \frac{2.8774}{2} \approx 1.4387$.

So, $\log x \approx 1.4387$, hence, $x = 10^{1.4387} \approx 27.460$. That is, $\sqrt{754} \approx 27.460$.

Example 2: Solve $4^x = 7^{x-3}$ for *x*.

Solution: $4^x = 7^{x-3} \xrightarrow[\text{property}]{\text{apply}} \log 4^x = \log 7^{x-3} \rightarrow x\log 4 = (x-3)(\log 7)$

$\xrightarrow[\text{on the right}]{\text{distribute}} x\log 4 = x\log 7 - 3\log 7 \xrightarrow[\text{calculator}]{\text{by}} 0.6021x = 0.8451x - 3(0.8451)$

$\xrightarrow[\text{with algebra}]{\text{proceed}} 3(0.8451) = 0.8451x - 0.6021x \rightarrow 2.5353 = 0.243x$, so $x \approx 10.433$

The technique of taking the log (either common, natural, or other) of both sides of an equation is used often to solve for unknown exponents, as happens with compounding of interest (see *ENGINEERING ECONOMICS* on page 138).

Arithmetic Sequence

An arithmetic sequence (also called an arithmetic progression) is a sequence of numbers in which each term differs from the preceding one by a fixed amount, called the *common difference*, d. Thus, 1, 3, 5, 7, etc. is an arithmetic sequence where the difference d is 2. Here, the consecutive terms of the sequence are increasing by 2. In the sequence 13, 10, 7, 4, etc., the difference is –3, and the sequence is decreasing. In any arithmetic progression (or portion of one):

a = first term of the sequence, also called the a_1 term
l = last term considered, also called a_n for the nth term
n = number of terms
d = common difference
S_n = sum of n terms

The formula for the last term is $l = a + (n-1)d$, or $a_n = a_1 + (n-1)d$. The sum of an arithmetic sequence with n terms is $S_n = \frac{n(a+l)}{2}$ or $\frac{n(a_1+a_n)}{2}$.

In these formulas, d is positive when the progression is increasing and negative when it is decreasing. When any three of the preceding five quantities are given, the other two can be found by the formulas in the table *Arithmetic Sequence Formulas* on page 35. Often, however, the desired quantity can be determined by working with the information given.

Example 1: In a given arithmetic progression, the first term is 5 and the last term 40, and the difference between terms is 7. To find the sum of the progression, first the number of terms has to be found. This is done by considering the difference between the first and last: 40 – 5 = 35. Dividing this by the difference between terms gives the number of intervals between the terms: 35 ÷ 7 = 5. Finally, adding 1 gives the number of terms in the sequence: $n = 5 + 1 = 6$. The sum of the sequence is:

$$S = \frac{n}{2}(a+l) = \frac{6}{2}(5+40) = 3(45) = 135$$

Geometric Sequence

A geometric sequence or progression is a sequence of numbers in which each term is derived by multiplying the preceding term by a constant multiplier called the *ratio*. When this ratio is greater than 1, the progression is increasing; when less than 1, it is decreasing. Thus, the sequence 2, 6, 18, 54, etc. is an increasing geometric sequence with a ratio of 3, and the sequence 24, 12, 6, etc. is a decreasing sequence with a ratio of 1⁄2.

In any geometric progression (or part of progression):

a = first term of the sequence
l = last (or nth) term of the sequence
n = number of terms
r = ratio of the progression
S_n = sum of n terms

The general formulas for the nth term: $l = ar^{n-1}$ and $S = \frac{rl - a}{r - 1}$

When any three of the preceding five quantities are given, the other two can be found by the formulas in the table *Geometric Sequence Formulas* on page 36. Geometric progressions are used for finding the successive speeds in machine tool drives, and in interest calculations.

Example 2: The lowest speed of a lathe is 20 rpm. The highest speed is 225 rpm. There are 18 speeds. Find the ratio between successive speeds.

$$\text{Ratio } r = \sqrt[n-1]{\frac{l}{a}} = \sqrt[17]{\frac{225}{20}} = \sqrt[17]{11.25} = 1.153$$

Arithmetic Sequence Formulas

To Find	Given	Use Formula
a	d l n	$a = l-(n-1)d$
	d n S	$a = \frac{S}{n} - \frac{n-1}{2} \times d$
	d l S	$a = \frac{d}{2} \pm \frac{1}{2}\sqrt{(2l+d)^2 - 8dS}$
	l n S	$a = \frac{2S}{n} - l$
d	a l n	$d = \frac{l-a}{n-1}$
	a n S	$d = \frac{2S-2an}{n(n-1)}$
	a l S	$d = \frac{l^2-a^2}{2S-l-a}$
	l n S	$d = \frac{2nl-2S}{n(n-1)}$
l	a d n	$l = a+(n-1)d$
	a d S	$l = -\frac{d}{2} \pm \frac{1}{2}\sqrt{8dS+(2a-d)^2}$
	a n S	$l = \frac{2S}{n} - a$
	d n S	$l = \frac{S}{n} + \frac{n-1}{2} \times d$
n	a d l	$n = 1 + \frac{l-a}{d}$
	a d S	$n = \frac{d-2a}{2d} \pm \frac{1}{2d}\sqrt{8dS+(2a-d)^2}$
	a l S	$n = \frac{2S}{a+l}$
	d l S	$n = \frac{2l+d}{2d} \pm \frac{1}{2d}\sqrt{(2l+d)^2-8dS}$
S	a d n	$S = \frac{n}{2}[2a+(n-1)d]$
	a d l	$S = \frac{a+l}{2} + \frac{l^2-a^2}{2d} = \frac{a+l}{2d}(l+d-a)$
	a l n	$S_n = \frac{n(a+l)}{2}$
	d l n	$S = \frac{n}{2}[2l-(n-1)d]$

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